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Alexei Kotov, Camille Laurent-Gengoux, Vladimir Salnikov

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ALEXEI KOTOV, CAMILLE LAURENT-GENGOUX, AND VLADIMIR SALNIKOV

Abstract. Following recent results of A.K. and V.S. on $\mathbb{Z}$-graded manifolds, we give several local and global normal forms results for Q-structures on those, i.e. for differential graded manifolds. In particular, we explain in which sense their relevant structures are concentrated along the zero-locus of their curvatures, especially when the negative-part is of Koszul-Tate type. We also give a local splitting theorem.

Introduction

This article is the sequel to [20] where normal forms of $\mathbb{Z}$-graded manifolds were studied and where the analogue of the Batchelor's theorem was proven. We now equip a $\mathbb{Z}$-graded manifold with a degree $+1$ self-commuting vector field Q, thus making it a differential graded (DG) manifold, also called Q-manifold ([30]). The purpose of this paper is to provide several normal form results in this setting. Notice that none of these "normal forms" are linearization results, which is an entirely different subject.
The paper is organized as follows. In Section 1, we give some precise definitions and fix some usual notations related to graded manifolds. In particular, we proceed with the description of projective systems of graded algebras (recapitulated in Appendix A) which we specialize to the $\mathbb{Z}$-graded structure sheaves. Section 2 is devoted to the idea that "outside the zero locus of their curvatures, ($\mathbb{Z}^*$-graded) Q-manifolds can be made trivial". A more precise statement is that, on any open subset where the curvature κ is different from zero at all points, the dual $\mathbb{Z}^*$-graded Lie ∞-algebroid can be chosen to have all k-ary bracket equal to zero, except for the 0-ary bracket given by the nowhere vanishing curvature κ. We consider $\mathbb{Z}^*$-graded manifolds for convenience, the difference with the $\mathbb{Z}$-graded case is rather technical, we will make a remark on it in Section 1.
In Section 3 we first recall the standard notion of Koszul–Tate resolution, which are examples of negatively graded Q-manifolds. Then we construct two structures on the zero locus $\{\kappa = 0\}$ of a Q-manifold that are independent of a choice of a splitting: a positively graded Q-structure on the zero locus $\{\kappa = 0\}$ and a negatively graded Q-manifold. We eventually show that Q-manifolds whose negative part is of Koszul-Tate type are entirely encoded by this positively graded Q-structure on the zero locus.
Last, in Section 4, we choose a point in the zero locus (on which leaves of the anchor map are well defined) and give a splitting theorem: near a leaf L in the zero locus, a Q-manifold is the direct product of the standard $T[1]L$ and a transverse Q-manifold. In the process, we also give some counter-examples to "naive beliefs" about the anchor maps of a Q-manifold. We conclude by mentioning some perspectives and potential applications.

Key words and phrases. $\mathbb{Z}$-graded manifolds, dg-manifolds, $\mathbb{Q}$-structures, Lie ∞-algebroids, normal forms, splitting theorems.

1. Notations and preliminaries

For a $\mathbb{Z}$-graded vector space $\mathcal{S}$, we denote by $\mathcal{S}_i$ the vector subspace of elements of degree i. Also, $\mathcal{S}_{\leq a}$ stands for $\oplus_{i\leq a}\mathcal{S}_i$ and $\mathcal{S}_{\geq a}$ stands for $\oplus_{i\geq a}\mathcal{S}_i$.

1.1. **$\mathbb{Z}^*$-graded manifolds.** Let us first recall the definition of $\mathbb{Z}^*$-graded manifolds. In what follows, we assume (non-graded) manifolds to be real and smooth, (graded) algebras to be real and unital. We start with an important definition: a filtration which is used throughout the paper.

Definition 1.1. *Let $\mathcal{S} = \oplus_{j\in\mathbb{Z}}\mathcal{S}_j$ be a $\mathbb{Z}$-graded commutative*[1] *algebra. We call* negative filtration *the filtration*

$$\mathcal{S} = F^0\mathcal{S} \supset F^1\mathcal{S} \supset \cdots \supset F^i\mathcal{S} \supset \dots$$

where, for all $i \geq 1$, $F^i\mathcal{S}$ is the ideal of $\mathcal{S}$ generated by elements of degree less or equal to $-i$, i.e.

$$\begin{array}{rclcl} F^0\mathcal{S} &=& \mathcal{S} && \\ F^1\mathcal{S} &=& \mathcal{S}\cdot\mathcal{S}_{\leq -1} &=& \mathcal{S}\cdot(\mathcal{S}_{-1}\oplus\mathcal{S}_{-2}\oplus\cdots) \\ F^2\mathcal{S} &=& \mathcal{S}\cdot\mathcal{S}_{\leq -2} &=& \mathcal{S}\cdot(\mathcal{S}_{-2}\oplus\mathcal{S}_{-3}\oplus\cdots) \\ &\vdots&&& \end{array}$$

When a filtration by ideals of an algebra is given, this allows to consider the projective limit $lim_{i\to\infty}\mathcal{S}/F^i\mathcal{S}$. For graded algebras, one has to consider the graded projective limit, see Appendix A or [20]: it consists in taking, for all $j \in \mathbb{Z}$, the projective limit $lim_{i\to\infty}\mathcal{S}_j/F^i\mathcal{S}_j$, where $F^i\mathcal{S}_j$ stands for elements in degree j in $F^i\mathcal{S}$. Altogether, these projective limits form a graded algebra. This notion permits to define the graded manifolds[2] we are interested in.

Definition 1.2 ([20])**.** *A* $\mathbb{Z}^*$-graded manifold *is a pair $M = (M_0, \mathcal{O})$, where M_0 is a smooth manifold (referred to as* base manifold*) and $\mathcal{O} = \oplus_{i\in\mathbb{Z}}\mathcal{O}_i$ is a sheaf of $\mathbb{Z}$-graded commutative algebras (whose sections are referred to as* functions*) such that each point of M_0 has a neighborhood $U \subset M_0$ over which $\mathcal{O}(U)$ is isomorphic to $\Gamma(\tilde{S}(\oplus_{i\in\mathbb{Z}^*}V_i))$, where each V_i is a vector bundle over U whose sections are considered to be of degree i. Last, "$\Gamma(\tilde{S}(\oplus_{i\in\mathbb{Z}^*}V_i))$" stands for the graded projective limit*[3] *of $\Gamma(S(\oplus_{i\in\mathbb{Z}^*}V_i))$ with respect to its negative filtration.*

Example 1.3. *There are many natural situations where positive and negative degrees appear simultaneously. This includes for instance:*

- *cotangent bundles to $\mathbb{Z}_{\geq 0}$-graded manifolds, in particular, for any vector bundle $E \to M$ seen as a graded manifold with $\mathcal{O}(U)_k = \Gamma_U(\wedge^k E)$, its cotangent bundle is the graded manifold over T^*M with the algebra of functions $\mathcal{O}_k := \bigoplus_{i-j=k}\Gamma(\wedge^i E \oplus \wedge^j E^*)$, E and E^* being the pull-back of E and F to T^*M;*
- *mapping spaces between $\mathbb{Z}_{\geq 0}$-graded manifolds;*

[1] Throughout the paper by commutative we mean supercommutative, with the parity which appears in the Koszul sign rule given by the $\mathbb{Z}$ degree modulo 2.

[2] See Remark 1.8 for the relation between this definition with the ones given by Theodore Voronov.

[3] We hesitated between the terminologies "graded projective limit" or "graded completion", since the graded projective limit is also the completion with respect to the topology induced by the graded filtration. Also, there is a strict inclusion $\Gamma(S(\oplus_{i\in\mathbb{Z}^*}V_i)) \hookrightarrow \Gamma(\tilde{S}(\oplus_{i\in\mathbb{Z}^*}V_i))$.

- *graded groups determined by Harish-Chandra pairs, see* [21].

More examples will appear at the end of Section 3 where infinitesimal symmetries of affine varieties will be dealt with the help of Koszul-Tate resolutions.

A $\mathbb{Z}^*$-graded manifold whose grading in fact only goes from 0 to $-\infty$ (resp. 0 to $+\infty$) shall be referred to as a *negatively-graded manifold* (resp. *positively-graded manifold*). In these cases, the grading of the vector bundle $V_\bullet$ goes from -1 to $-\infty$ and from $+1$ to $+\infty$ respectively. Also, in these two situations, it is not necessary to consider the graded projective limit, since both $\Gamma(S(\oplus_{i\leq -1}V_i))$ and $\Gamma(S(\oplus_{i\geq 1}V_i)$ are equal to their graded projective limits with respect to their negative filtrations.

Remark 1.4. Definition 1.2 is explained in detail in [20]: for this paper to be self-consistent and for further use, we recollect some necessary facts about filtrations and their graded projective limits in what follows and in Appendix A.
Notice that Definition 1.2 potentially allows a function on a graded manifold to be a sum of infinitely many terms. We refer to [20] for the subtle point of knowing which infinite sums are allowed in the projective limits: for a family $v_i \in \Gamma(V_i)$, the sums $\sum_{i\geq 1} v_i v_{-i}$, $\sum_{i\geq 1} v_{2i} v_{-2}^i$, or $\sum_{i\geq 1} v_2^i v_{-2}^i$ make sense and are of degree 0, but the sum $\sum_{i\geq 1} v_{-i}$ for non-zero v_{-i} does not make sense. □

Remark 1.5. We write "$\mathbb{Z}^*$-graded" instead of "$\mathbb{Z}$-graded" to insist on the natural assumptions that there are no generators of $\mathcal{O}$ of degree 0 which is not a coordinate function on M_0. For instance, Kapranov dg-manifolds [24, 25] are not $\mathbb{Z}^*$-graded manifolds; non-linear Lie algebroids [36] are also excluded.
Note, however, that the difference between the $\mathbb{Z}^*$- and $\mathbb{Z}$-graded manifolds in terms of [20] is not really conceptual. Certainly, including a degree 0 vector space to the base or to the direct sum in $\tilde{S}(\dots)$ will produce some *technical* ambiguities for the set of allowed degree 0 generators of the functional spaces, which will fade out after proper redefinition of their filtrations.
For $\mathbb{Z}^*$-graded manifolds, in contrast to the positively-graded or negatively-graded cases, we do not have an isomorphism $\mathcal{C}^\infty(M_0) \simeq \mathcal{O}_0$. For instance, the product of a function in $\mathcal{O}_p$ with a function in $\mathcal{O}_{-p}$ may very well produce a non-zero function: it then belongs to $\mathcal{O}_0$ but can not be considered as an element in $\mathcal{C}^\infty(M_0)$. There is even no canonical inclusion $\mathcal{C}^\infty(M_0) \hookrightarrow \mathcal{O}_0$, but there is a natural projection $\mathcal{O}_0 \to \mathcal{C}^\infty(M_0)$, which corresponds to the inclusion $M_0 \hookrightarrow M$. □

Remark 1.6. There is a unique filtration on $\mathcal{O}$ induced by the negative filtration of the symmetric algebras that appear in Definition 1.2. We still denote it by $(F_i\mathcal{O})_{i\geq 0}$ and call it the *negative filtration of* $\mathcal{O}$. Notice that elements in $F^i\mathcal{O}$ may be of any degree, although its generators have degree less or equal to $-i$. Also, notice that $\cap_{i\geq 0} F^i\mathcal{O} = \{0\}$. □

Remark 1.7. A $\mathbb{Z}^*$-graded manifold $(M_0, \mathcal{O})$ is complete with respect to the topology on $\mathcal{O}$ given by the negative filtration of Remark 1.6, see [20]. □

Remark 1.8. Our definition of graded manifolds matches more or less the one given by Theodore Voronov in Section 4 of [34], which the author himself links to previously introduced objects by Ševera [31] or Kontsevich [18]. We claim to be more precise about "allowed" or "forbidden" infinite sums, which obliged us to impose some restrictions on degree 0 variables that [34] do not require. Our definition matches also those given in Section 2.2.1 [37] when finitely many of the V_i are non-zero. Otherwise, [37] allows (in the model that is called (iii) in Section 2.1.1) more infinite sums considering formal sums in all non-zero degree variables, which is the completion with respect to the ideal generated by all functions of non-zero degree (like the starting point in the Appendix of [15]), and not the finer completion with respect to the filtration $F^i\mathcal{S}$, that we consider here. □

According to [20], there are natural sheaves of graded ideals in $\mathcal{O}$:

(1) the ideal $\mathcal{I}_+$ generated by $\oplus_{i\geq 1}\mathcal{O}_i$.
(2) the ideal $\mathcal{I}_- = F^1\mathcal{O}$ generated by $\oplus_{i\leq -1}\mathcal{O}_i$, called *ideal of functions vanishing on the zero section.*
(3) The ideal $\mathcal{I} = \mathcal{I}_+ + \mathcal{I}_-$.

Let us consider the quotient of $\mathcal{O}$ by these three ideals:

(1) The quotient $(M_0, \mathcal{O}/\mathcal{I}_+)$ is a graded manifold with grading now ranging from 0 to $-\infty$, i.e. a negatively-graded manifold, that we call the *negative part of* $(M_0, \mathcal{O})$.
(2) The quotient $(M_0, \mathcal{O}/\mathcal{I}_-)$ is a graded manifold with grading now ranging from 0 to $+\infty$, i.e. a positively-graded manifold, that we call the *positive part of* $(M_0, \mathcal{O})$.
(3) The quotient $(M_0, \mathcal{O}/\mathcal{I})$ is simply the smooth manifold M_0 equipped with its sheaf $C^\infty(M_0)$ of smooth functions (and, in particular, is concentrated in degree 0).

To a graded manifold $M = (M_0, \mathcal{O})$, one can associate (canonically) a family $(E_i)_{i\in\mathbb{Z}^*}$ of vector bundles over M_0, as follows. The quotient space

$$\frac{\mathcal{I}}{\mathcal{I}^2} = \bigoplus_{i\in\mathbb{Z}^*} \frac{\mathcal{I}_i}{(\mathcal{I}^2)_i} \tag{1.1}$$

is a direct sum of projective $\mathcal{C}^\infty(M_0)$-modules, hence by Serre–Swan theorem, there exists for all $i \in \mathbb{Z}^*$ a vector bundle E_i such that $\Gamma((E_i)^*) \simeq \mathcal{I}_{-i}/(\mathcal{I}^2)_{-i}$. We call $E_\bullet := \oplus_{i\in\mathbb{Z}^*} E_i$ the *canonical graded vector bundle of* $(M_0, \mathcal{O})$.

There exists in this context a precise equivalent of the so-called Batchelor's theorem in supergeometry, which appeared[4] in the Ph.D. thesis of Batchelor [3].

Theorem 1.9 (*Batchelor's theorem*, [20] – Sections 3.3 and 4.2)**.** *Let $(M_0, \mathcal{O})$ be a $\mathbb{Z}^*$-graded manifold with canonical $\mathbb{Z}^*$-graded bundle $(E_i)_{i\in\mathbb{Z}^*}$. There exists an isomorphism of sheaves (called* splitting*):*

$$\mathcal{O} \simeq \Gamma\left(\tilde{S}(\oplus_{i\in\mathbb{Z}^*} E_i^*)\right).$$

[4]It was independently formulated by K. Gawedzki [10]. The result was also obtained, more or less simultaneously, by Berezin [5], and mentioned notes of courses given by Palamodov in Moscow at that time, as well as some additions by Palamodov on a posthumous edition of Berezin's work. Also, there are others Batchelor's theorems adapted to different contexts of graded manifolds. For $\mathbb{Z}_{\geq 0}$-graded manifold, the result belongs to Roytenberg (see e.g. [28]) and is formulated in terms of a tower of fibrations of graded manifolds. In the $\mathbb{Z}$-graded case a Batchelor-type theorem (called classification theorem for smooth graded manifolds) is given in [37], where it is treated as a particular case of the tubular neighborhood theorem, extending the same idea for supermanifolds ([33]), suggested in that context independently by Koszul [19], see also [14]. For the general $\mathbb{Z}$-graded case the result was independently reformulated more explicitly in [20], using a different type of filtration of the functional space, allowing thus to produce the normal form similar to Roytenberg's one and a functor from $\mathbb{Z}$-graded to $\mathbb{N}^2$-graded manifolds.

Here $\Gamma\left(\tilde{S}(\oplus_{i\in\mathbb{Z}^*}E_i^*)\right)$ *refers to the graded projective limit of* $\Gamma\left(S(\oplus_{i\in\mathbb{Z}^*}E_i^*)\right)$ *with respect to its negative filtration as in Definition 1.1.*

Remark 1.10. For negatively graded or positively graded manifolds, there is no need to consider the graded projective limit in Batchelor's theorem.

Remark 1.11. Notice that for every splitting, sections of $E_{-i}^* \equiv (E_{-i})^* = (E^*)_i$ become functions of degree $+i$ in $\mathcal{O}$. Also, notice that the bundle V_i is Definition 1.2 can be chosen to be the restriction to U of $(E_{-i})^*$. □

Remark 1.12. Although Batchelor's theorem claims that splittings exist, there is no canonical splitting in general. In contrast, the vector bundles $(E_i)_{i\in\mathbb{Z}^*}$ defined above are canonical. □

Once a splitting is chosen, many different notions of "degree" can be defined, beside the degree that $\mathcal{O}$ is equipped with by definition. More precisely, for a section $\alpha \in \Gamma(E^*)_i = \Gamma((E_{-i})^*)$, let us define three different degrees as follows:

$$\deg(\alpha) = i,\ \ \mathrm{pol}(\alpha) = 1,\ \ \deg_+(\alpha) = \begin{cases} \mathrm{i} & \text{for } i \geq 1 \\ 0 & \text{otherwise,} \end{cases},\ \ \deg_-(\alpha) = \begin{cases} -i & \text{for } i \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then these degrees extend by multiplicativity to $\Gamma(S(\oplus_{i\in\mathbb{Z}^*}E_i^*))$. To avoid confusion, the degree deg will be called the *total degree*, sometimes referred to as the *ghost degree*. It coincides with the degree that $\mathcal{O}$ is initially equipped with. This degree is responsible[5] for all the commutation relations, i.e. the Koszul sign rule is defined by its reduction modulo 2. The degree $\deg_-$ (resp. $\deg_+$) is called the *negative degree* (resp. *positive degree*) and plays an important role. Also,

$$\deg = \deg_+ - \deg_-.$$

Last, pol is the *polynomial degree* (sometimes referred to as *arity*) that counts the number of sections in a product.

Example 1.13. Concretely, for a section of the symmetric product $E_{-5}^* \odot E_4^* \odot E_7^*$

- the total degree or ghost degree is $5 - 4 - 7 = -6$;
- the negative degree is $4 + 7 = +11$; Notice the "+" sign.
- the positive degree is $+5$;
- the polynomial degree is 3 (it is the product of three sections).

Remark 1.14. The negative degree is compatible with the filtration $F^i\mathcal{O}$ introduced above in the sense that $F^i\mathcal{O} = \{F \in \mathcal{O} \,|\, \deg_-(F) \geq i\}$. □

1.2. ***Q*-manifolds.** Let us now define Q-manifolds, that is equip a $\mathbb{Z}^*$-graded manifold with a differential.

Definition 1.15. *A vector field of degree k on a $\mathbb{Z}^*$-graded manifold $(M, \mathcal{O})$ is a degree k derivation of $\mathcal{O}$.*

[5]Recall that, we assume the parity to be given by the degree modulo 2 for simplicity of the presentation in this paper, but the constructions work for a more general convention on the relation of the total degree and the (super) parity, namely when each homogeneous vector subspace of a given degree is a superspace itself.

The space of vector fields of degree k shall be denoted as $\mathfrak{X}_k(\mathcal{O})$. The graded vector space of all vector fields:

$$\mathfrak{X}_\bullet(\mathcal{O}) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{X}_k(\mathcal{O}),$$

forms a graded Lie algebra when equipped with the graded commutator $[\cdot,\cdot]$.

Definition 1.16. *A $\mathbb{Z}^*$-graded Q-manifold is a triple $(M_0, \mathcal{O}, Q)$, with $M = (M_0, \mathcal{O})$ a $\mathbb{Z}^*$-graded manifold and Q a degree $+1$ vector field which satisfies $[Q, Q] = 0$.*
For $M = (M_0, \mathcal{O})$ a positively graded manifold (resp. negatively graded manifold), we shall speak of a positively-graded Q-manifold (resp. negatively-graded Q-manifold)

Since the degree of Q is $+1$, we have $Q[\mathcal{I}_+] \subset \mathcal{I}_+$, so that Q induces a degree $+1$ derivation δ of the quotient $\mathcal{O}/\mathcal{I}_+$ which is by definition the sheaf of functions on a negatively graded manifold. This allows the following definition.

Definition 1.17. *We call the negatively graded Q-manifold $(M_0, \mathcal{O}/\mathcal{I}_+, \delta)$ the* negative part *of the Q-manifold $(M_0, \mathcal{O}, Q)$.*

Remark 1.18. The vector field Q^- is $\mathcal{C}^\infty(M_0)$-linear, i.e. it is a vertical vector field. □

1.3. **An algebraic generalization: Q-varieties over a commutative algebra.** Let $\mathcal{A}$ be an commutative algebra with unit (that may be thought as functions over an affine variety X_0 for instance). Definitions 1.2 and 1.16 admit a generalization:

Definition 1.19. *Let $I \subset \mathcal{C}^\infty(M_0)$ be an ideal. A positively graded variety (resp. Q-variety) over $\mathcal{C}^\infty(M_0)/I$ is a positively graded commutative algebra $\mathcal{K}_+$ (resp. positively graded commutative differential algebra $(\mathcal{K}_+, Q_+)$) that admits a splitting, i.e. an isomorphism*

$$\mathcal{K}_+ \simeq \Gamma_I(S(\oplus_{i\geq 1} E^*_{-i}))$$

for a family of vector bundles $(E_{-i})_{i\geq 1}$ over M_0. Here for any vector bundle $E \to M_0$,

$$\Gamma_I(E) := \Gamma(E) \otimes_{\mathcal{C}^\infty(M_0)} \mathcal{C}^\infty(M_0)/I.$$

Remark 1.20. There is no need to take graded projective limits in the definition above since every function of a given degree is necessarily polynomial with respect to non-zero degree variables.

1.4. **Duality Q-manifolds $\sim$ Lie ∞-algebroids.** Let $(M_0, \mathcal{O}, Q)$ be a Q-manifold. Once a splitting $\mathcal{O} \simeq \Gamma(\oplus_{i\in\mathbb{Z}} E^*_i)$ is given, Q can be dualized to a Lie ∞-algebroid, defined as follows.

Definition 1.21. [17, 8] *A $\mathbb{Z}^*$-graded Lie ∞-algebroid structure on a $\mathbb{Z}^*$-graded vector bundle is the data of:*

- *families indexed by $n \geq 1$ of vector bundle morphisms*

  $$\rho_n \colon S^n(\oplus_{i\in\mathbb{Z}^*} E_i)_{-1} \longrightarrow TM_0$$

  called n-anchor maps,
- *families of degree $+1$ maps:*

  $$\ell_n \colon S^n_{\mathbb{R}} \left(\Gamma(\oplus_{i\in\mathbb{Z}^*} E_i)\right)_k \longrightarrow \Gamma(E_{k+1})$$

  called n-bracket,

together with a section $\kappa \in \Gamma(E_{+1})$ called curvature *that satisfy the higher Jacobi and higher Leibniz identities (see e.g.* [29]*).*

Remark 1.22. It is not easy to attach a single name to the following proposition, based on a observation by Pavol Ševera [38], spelled out in the negative degree case in [7], and which can be proven using Voronov's derived brackets in [35]. □

Proposition 1.23. *There is a one-to-one correspondence between $\mathbb{Z}^*$-graded Lie ∞-algebroids structures on $\oplus_{i\in\mathbb{Z}^*}E_i \to M_0$ and Q-manifolds structures with sheaf of functions $\Gamma(\tilde{S}(\oplus_{i\in\mathbb{Z}^*}E_i^*))$.*

1.5. **Projective systems associated to graded manifolds.** In this section, we give a precise sense to the notion of the flow of a degree 0 vector field on a graded manifold. For the standard definitions of projective systems the reader is referred to Appendix A, while now we specialize the Proposition A.1 from there to the context we are interested in. Let $(M_0, \mathcal{O})$ be a $\mathbb{Z}^*$-graded Q-manifold over M_0 with the sheaf of functions $\mathcal{O}$. This sheaf of functions comes equipped with the (negative) filtration as in Definition 1.1, so that $A^i := \mathcal{O}/F^i\mathcal{O}$ is a projective system of graded algebras. Since $\cap_{i\in\mathbb{N}}F^i\mathcal{O} = \{0\}$, its graded projective limit A^∞ is canonically isomorphic to $\mathcal{O}$.
If a degree 0 vector field $\boldsymbol{v}$ such that $\boldsymbol{v}[\mathcal{O}] \subset F^n\mathcal{O}$ for some $n \geq 1$ is given, then for every $i \in \mathbb{N}$, the family of endomorphisms

$$\begin{array}{rcl} \mathcal{O}/F^i\mathcal{O} & \to & \mathcal{O}/F^i\mathcal{O} \\ f & \mapsto & \sum\limits_{k\geq 0} \frac{t^k}{k!}\boldsymbol{v}^k[f] \end{array}$$

is well-defined because the sum is finite, it is an algebra endomorphism for all $i \in \mathbb{N}$, and is a morphism of projective systems of algebras. We denote its projective limit by $e^{t\boldsymbol{v}}$. By construction, for all $s, t \in \mathbb{R}$ we have $e^{s\boldsymbol{v}}e^{t\boldsymbol{v}} = e^{(s+t)\boldsymbol{v}}$ and $e^{0\boldsymbol{v}} = \mathrm{Id}_{\mathcal{O}}$. As a consequence $e^{t\boldsymbol{v}}$ is a diffeomorphism of the graded manifold $(M_0, \mathcal{O})$.

Proposition 1.24. *Given a family $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ of degree zero vector fields on a graded manifold $(M_0, \mathcal{O})$ such that*

$$\boldsymbol{v}_n \colon \mathcal{O} \to F^n\mathcal{O}$$

the infinite composition $\bigcirc_{i\uparrow\in\mathbb{N}} e^{\boldsymbol{v}_i} \equiv \cdots \circ e^{\boldsymbol{v}_2} \circ e^{\boldsymbol{v}_1}$ is a diffeomorphism of the graded manifold $(M_0, \mathcal{O})$, well-defined in the sense of [20].

2. Q-manifolds with curvature

2.1. **Normal forms outside of the zero locus of the curvature.** For $(M_0, \mathcal{O}, Q)$ a $\mathbb{Z}^*$-graded Q-manifold[6], recall (Equation 1.1) that the associated canonical vector bundle E_{+1} (in fact its dual) is defined by applying the Serre-Swan theorem:

$$\Gamma(E_{+1}^*) = \frac{\mathcal{I}_{-1}}{(\mathcal{I}^2)_{-1}}$$

where $\mathcal{I} \subset \mathcal{O}$ is the ideal of functions vanishing on the zero section.

[6]We present the results of this section for Q-manifolds with a smooth base. All results in section 2 extend to Q-manifolds over affine varieties or Q-manifolds over Stein varieties. This may no longer be true for the results of Section 3.

Definition 2.1. *The composition*

$$\mathcal{I}_{-1} \xrightarrow{Q} \mathcal{O}_0 \longrightarrow \mathcal{C}^\infty(M_0) \simeq \frac{\mathcal{O}_0}{\mathcal{I}_0}$$

is $\mathcal{C}^\infty(M_0)$-linear and contains $(\mathcal{I}^2)_{-1}$ in its kernel. It is therefore given by the contraction with a canonical section of E_{+1} that we call the curvature of the Q-manifold M *and* denote by κ.

Equivalently, since $\mathcal{I}_0 = F^1\mathcal{O}_0$, $\mathcal{I}_{-1} = F^1\mathcal{O}_{-1}$, and $(\mathcal{I}^2)_{-1} = F^2\mathcal{O}_{-1}$ the curvature is defined by the following commutative diagram, whose horizontal lines are exact:

$$\begin{array}{ccccc} F^2\mathcal{O}_{-1} & \hookrightarrow & F^1\mathcal{O}_{-1} & \twoheadrightarrow & \Gamma(E^*_{+1}) \\ \downarrow Q & & \downarrow Q & & \vdots\, \mathfrak{i}_\kappa \\ F^1\mathcal{O}_0 & \hookrightarrow & \mathcal{O}_0 & \twoheadrightarrow & \mathcal{C}^\infty(M_0). \end{array}$$

Remark 2.2. The previous description of the curvature, although abstract, implies that it is a canonical notion, but it can be described in a more explicit manner, upon choosing a splitting. The polynomial degree is then well-defined, and $\mathfrak{i}_\kappa$ is the only component of Q of polynomial degree -1.

$$Q = \mathfrak{i}_\kappa + \sum_{i \geq 0} Q^{[i]}$$

where $Q^{[i]}$ is the component of polynomial degree i of Q. Also, after having chosen a splitting and local coordinates:

$$Q = \sum_{i=1}^{\mathrm{rk}(E_{+1})} \tilde{\kappa}_i(x) \frac{\partial}{\partial \eta_i} + \sum_{j=1}^{\dim(M_0)} f_j \frac{\partial}{\partial x_j} + \sum_{i \in \mathbb{Z}\setminus\{0,1\}} \sum_{j=1}^{\mathrm{rk}(E_i)} g_{i,j} \frac{\partial}{\partial \theta_{i,j}}. \tag{2.2}$$

Here the x_i's are the variables in the base manifold, the η_i's are the degree -1 variables, the $\theta_{i,j}$'s are the degree j variables for $j \neq 0, -1$, the functions $\tilde{\kappa}_i(x) \in \mathcal{O}_0$ are functions whose projection in $\mathcal{C}^\infty(M_0)$ are the components of the section κ, $f_j \in \mathcal{O}_1$, and $g_{i,j} \in \mathcal{O}_{1-i}$. □

It is well-known [27] that on a supermanifold of dimension (n, p), around every point where a self-commuting odd vector field Q does not vanish on the zero section, there exist local coordinates $(x_1, \ldots, x_n, \eta_1, \ldots, \eta_p)$ such that

$$Q = \tfrac{\partial}{\partial \eta_1}. \tag{2.3}$$

Below is the equivalent of this statement for the $\mathbb{Z}^*$-graded case.

Proposition 2.3. *Let $(M_0, \mathcal{O}, Q)$ be a $\mathbb{Z}^*$-graded Q-manifold with associated canonical bundles $(E_i)_{i\in\mathbb{Z}^*}$ over M_0. Over every open set $U \subset M_0$ over which the curvature $\kappa \in \Gamma(E_1)$ is different from zero at every point, there is a splitting $\mathcal{O}(U) \simeq \Gamma\left(\tilde{S}\left(\oplus_{i\in\mathbb{Z}^*} E_i^*\right)\right)$ under which*

$$Q = \mathfrak{i}_\kappa,$$

i.e. the degree $+1$ vector field Q is given by the contraction with the curvature.

Remark 2.4. In the situation when there is duality (in the sense of Section 1.4), Proposition 2.3 may be restated as follows: every open subset on which the curvature $\kappa \in \Gamma(E_{+1})$ is different from zero at every point admits a dual $\mathbb{Z}^*$-graded Lie ∞-algebroid for which all the brackets $(\ell_k)_{k\geq 1}$ are equal to zero except for the 0-ary bracket (which is κ). Notice that the proposition holds for the whole M_0 zero locus of κ. Also, Proposition 2.3 immediately implies the existence of local coordinates as in Equation (2.2) such that Q takes the form (2.3). □

The proof of Proposition 2.3 goes through the next three lemmas (see Definition 1.17 for the negative part δ of the vector field Q).

Lemma 2.5. *There exists a degree -1 function $\alpha \in F^1\mathcal{O}$ such that $\delta(\alpha) = 1 \in \mathcal{O}$.*

Proof. Take any splitting $\mathcal{O}(U) \simeq \Gamma\left(\tilde{S}(\oplus_{i\in\mathbb{Z}^*} E_i^*)\right)$. Since the curvature κ is a nowhere vanishing section of E_{+1}, there exists $\alpha \in \Gamma(E_{+1}^*) \subset F^1\mathcal{O}$ such that $\langle \kappa, \alpha\rangle = 1$. We then have $Q(\alpha) = \langle \kappa, \alpha\rangle + F = 1 + F$ for some function $F \in F^1\mathcal{O}_0 = \mathcal{O}_0 \cap \mathcal{I}_- = \mathcal{O}_0 \cap \mathcal{I}_+$. As a consequence, $\delta(\alpha) = 1$. ■

Lemma 2.6. *There exists a splitting $\mathcal{O}(U) = \Gamma_U\left(\tilde{S}(\oplus_{i\neq 0} E_i^*)\right)$ such that $Q = \delta$.*

Proof. The choice of a splitting $\mathcal{O}(U) \simeq \Gamma\left(\tilde{S}(\oplus E^*)\right)$ allows to decompose functions and vector fields according to their negative degree, and any function of given degree decomposes as a sum $f = \sum_{n\geq 0} f^{(n)}$ with $f^{(n)}$ a function of negative degree n ($deg_-(f^{(n)}) = n$). For a degree $+1$ vector fields R, we have:

$$R = \sum_{i\geq -1} R^{(n)}$$

with $R^{(n)}$ a vector field of negative degree n. Notice that, for instance, $\delta = Q^{(-1)}$.
We construct by induction a sequence $\Phi_n = e^{v_n}$ (starting at $n = 1$) of graded manifold isomorphisms that satisfy the following conditions:

(1) v_n is a vector field such that $v_n : \mathcal{O} \to F^n\mathcal{O}$ for all $n \in \mathbb{N}$ (i.e. $v_n^{(i)} = 0$ for $i < n$).
(2) the push-forward Q_n of the vector field Q by $\Phi_n \circ \cdots \circ \Phi_1$ is of the form:

$$Q_{n+1} = Q^{(-1)} + Q_{n+1}^{(n+1)} + \cdots$$

The sequence is constructed as follows: $Q_0 = Q$ and at each step we choose $v_{n+1} = -\alpha Q_n^{(n)}$, with α as in Lemma 2.5. It follows from $[Q_n, Q_n] = 0$ that $[Q^{(-1)}, Q_n^{(n)}] = 0$. As a consequence, the push-forward vector of Q_n by e^{v_n}, i.e. the derivation:

$$e^{-v_n} Q_n e^{v_n} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathrm{ad}_{v_n}^k Q \text{ (all sums are finite for a given negative degree)}$$

is given (up to components of negative degree $\geq n+1$) by

$$Q_n + [Q_n, v_n] = Q^{(-1)} + Q_n^{(n)} - [Q^{(-1)}, \alpha Q_n^{(n)}] = Q^{(-1)} + Q_n^{(n)} - Q_n^{(n)} = Q^{(-1)}.$$

The hereby constructed sequence satisfies the required assumption. We then apply Proposition 1.24 to construct the infinite composition $\Psi := \bigcirc_{i\uparrow\geq 1} e^{v_i}$. By construction, the push-forward of Q through Ψ is δ, which completes the proof. ■

Lemma 2.7. *There exists a splitting* $\mathcal{O}(U) = \Gamma(\tilde{S}(\oplus_{i\neq 0}E_i^*))$ *such that* $Q^{(-1)} = \mathfrak{i}_\kappa$.

Proof. The proof consists in repeating the steps of the proof of Lemma 2.6, by using now the polynomial degree, which is well-defined in the negative part. We write

$$Q^{(-1)} = \mathfrak{i}_\kappa + Q^{[0]} + Q^{[1]} + \cdots ,$$

where $[i]$ now stands for the polynomial degree. We then transport $Q^{(-1)}$ through $e^{\alpha Q^{[0]}}$. Since $[\mathfrak{i}_\kappa, Q^{[0]}] = 0$, the vector field obtained in such a way is now of the form:

$$Q_1^{(-1)} = \mathfrak{i}_\kappa + Q_1^{[1]} + Q_1^{[2]} + \cdots ,$$

for new $(Q_1 - \mathfrak{i}_\kappa)$ of polynomial degree ≥ 1. We then construct recursively a collection of isomorphisms of the graded manifold M that satisfy the requirements of Proposition 1.24: since we only use negative variables at this point, the ideal of elements of polynomial degree k in negative variables is included in $F^k\mathcal{O}$ (cf. to be more precise [20])). Their infinite composition intertwines $Q^{(-1)}$ with $\mathfrak{i}_\kappa$. ■

Proof. (of Proposition 2.3) The statement follows from Lemmas 2.6 and 2.7 above: Lemma 2.6 constructs an isomorphism of graded manifold under which Q becomes its negative part part δ, and Lemma 2.7 constructs an isomorphism of graded manifold under which δ becomes $\mathfrak{i}_\kappa$. ■

Corollary 2.8. *Let* $(M_0, \mathcal{O}, Q)$ *be a* $\mathbb{Z}^*$*-graded* Q*-manifold. On every open set* $U \subset M_0$ *over which the curvature* $\kappa \in \Gamma(E_1)$ *is different from zero at every point, the cohomology of* $(\mathcal{O}(U), Q)$ *is zero in every degree.*

Proof. The statement follows from the easily-checked fact that multiplication by the function $\alpha \in \Gamma(E_{+1}^*)$ defined in Lemma 2.5 is a contracting homotopy for $Q = \mathfrak{i}_\kappa$. ■

2.2. Geometry of the zero locus of the curvature of a Q-manifold.

Consider a Q-manifold $(M_0, \mathcal{O}, Q)$, with associated canonical vector bundles $(E_i)_{i\in\mathbb{Z}^*}$ and curvature $\kappa \in \Gamma(E_{+1})$ (see Definition 2.1).

Definition 2.9. *We call the* zero locus ideal of $\mathcal{O}$ *the image of*

$$\mathfrak{i}_\kappa \colon \Gamma(E_{+1}^*) \to \mathcal{O}$$

and we denote it by $\langle\kappa\rangle$. *We call* functions on the zero locus *the quotient algebra* $\mathcal{O}/\langle\kappa\rangle$.

The space $\mathcal{I}_- + \mathcal{O}Q[\mathcal{I}_-] \subset \mathcal{O}$ is both an ideal of $\mathcal{O}$ and stable by Q, so that the latter induces a derivation Q^+ of the quotient

$$\mathcal{K}_+ := \frac{\mathcal{O}}{\mathcal{I}_- + \mathcal{O}Q[\mathcal{I}_-]}. \tag{2.4}$$

The pair $(\mathcal{K}_+, Q_+)$ is a differential graded algebra (with grading from 0 to $+\infty$). Here is an important result.

Proposition 2.10. *The differential graded algebra* $(\mathcal{K}_+, Q_+)$ *of a* $\mathbb{Z}^*$*-graded* Q*-manifold* $(M_0, \mathcal{O}, Q)$ *is a positively graded* Q*-variety over the algebra* $\mathcal{C}^\infty(M_0)/\langle\kappa\rangle$ *of functions on the zero locus and there is a splitting*

$$\mathcal{K}_+ \simeq \Gamma_{\langle\kappa\rangle}\left(S(\oplus_{i\geq 1}E_{-i}^*)\right). \tag{2.5}$$

Here $\langle\kappa\rangle$ *is the zero-locus ideal and* $\Gamma_{\langle\kappa\rangle}(E) = \Gamma(E) \otimes_{\mathcal{C}^\infty(M_0)} \mathcal{C}^\infty(M_0)/\langle\kappa\rangle$ *for every vector bundle* $E \to M$*.*

Before proving the previous statement, let us notice that it allows to introduce the following definition.

Definition 2.11. *Let* $(M_0, \mathcal{O}, Q)$ *be a* $\mathbb{Z}^*$*-graded* Q*-manifold. We call* zero locus positively graded Q-variety *the positively graded* Q*-variety with sheaf of functions* $\mathcal{K}_+$ *and differential* Q_+ *as in Equation* (2.4)*.*

We now prove Proposition 2.10. We start with a lemma.

Lemma 2.12. *For any* $\mathbb{Z}^*$*-graded* Q*-manifold* $(M_0, \mathcal{O}, Q)$ *with curvature* κ*:*

$$\mathcal{O}Q[\mathcal{I}_-] + \mathcal{I}_- = \langle\kappa\rangle\mathcal{O} + \mathcal{I}_-.$$

Proof. For any $\alpha \in \Gamma_{E^*_{+1}}$:

$$\langle\kappa, \alpha\rangle = Q[\alpha] + \sum_{i\geq 1} F_i G_i$$

where $F_i, G_i \in \mathcal{O}$ are functions of degree $-a_i$ and $+a_i$ respectively for some $a_i \geq 1$ (the sum might be infinite). This proves the inclusion

$$\langle\kappa\rangle\,\mathcal{O} + \mathcal{I}_- \subset \mathcal{O}Q[\mathcal{I}_-] + \mathcal{I}_-.$$

The converse inclusion is straightforward. ■

Proof. (of Proposition 2.10) As a consequence of Lemma 2.12 above, the graded algebra morphism

$$\Gamma\left(S\left(\oplus_{i\geq 1} E^*_{-i}\right)\right) \to \mathcal{K}_+$$

is surjective, so that the following sequence is exact:

$$0 \to \langle\kappa\rangle\ \Gamma\left(S\left(\oplus_{i\geq 1} E^*_{-i}\right)\right) \to \Gamma\left(S\left(\oplus_{i\geq 1} E^*_{-i}\right)\right) \to \mathcal{K}_+ \to 0.$$

Consequently:

(1) the degree of elements in $\mathcal{K}_+$ is non-negative by construction,
(2) degree 0-elements can be identified with $\mathcal{O}(M_0)/\langle\kappa\rangle$,
(3) for $k \geq 1$, degree $+k$ elements are elements of degree k in the symmetric algebra (over $\mathcal{O}(M_0)/\langle\kappa\rangle$) of $\oplus_{i\geq 1}\Gamma(E^*_{-i}) \otimes \mathcal{O}(M_0)/\langle\kappa\rangle$.

This yields the isomorphism of projective $\mathcal{O}(M_0)/\langle\kappa\rangle$-module in Equation (2.5). ■

Remark 2.13. For $(M_0, \mathcal{O}, Q)$ be a $\mathbb{Z}^*$-graded Q-manifold with splitting, Q can be decomposed by the negative degree as an infinite sum:

$$Q = q_{-1} + q_0 + \cdots + q_i + \dots$$

with q_i a degree $+1$ vector field of negative degree i for $i \geq -1$. Then, it is easy to see that q_{-1} induces the negative part of the Q-manifold and that q_0 (which commutes with q_{-1}, hence induces a derivation of $\mathcal{K}_+$) induces the differential Q_+ of the zero locus positively graded Q-variety. □

Remark 2.14. As explained in [23], when the ideal κ is the ideal of functions vanishing on a submanifold $X \subset M_0$, then the distribution $\mathcal{D} := \rho_1(\Gamma(E_{-1}))$ is made of vector fields tangent to X and its restriction to X is involutive on X. This singular foliation on the submanifold X is the basic singular foliation of the positively graded Q-manifold $(\mathcal{K}_+, Q_+)$. The same conclusion holds when X is a singular subset, provided that vector fields on X can be defined in a appropriate manner (e.g.: an affine variety). □

3. Koszul-Tate resolution and vector fields on the zero locus positively graded Q-variety

Recall that for any vector bundle $E \to M_0$ and any ideal $I \subset \mathcal{C}^\infty(M_0)$, we use the following notation:

$$\Gamma_I(E) := \Gamma(E) \otimes_{\mathcal{C}^\infty(M_0)} \mathcal{C}^\infty(M_0)/I. \tag{3.6}$$

If I is the vanishing ideal of a submanifold $X_I \subset M_0$ (i.e. X_I is the zero locus of I), then $\Gamma_I(E)$ is simply the space of sections of the restriction of E to X_I.

3.1. Koszul-Tate resolutions.

Let M_0 be a smooth manifold. We recall the usual definition of a Koszul-Tate resolution of an ideal.

Definition 3.1. *A* Koszul-Tate resolution[7] *of an ideal $I \subset \mathcal{C}^\infty(M_0)$ is a Q-manifold $(M_0, \mathcal{O}_-, \delta)$ which*

(1) *is negatively graded, i.e. $\mathcal{O}_- = \oplus_{i \leq 0} \mathcal{O}_i$, with $\mathcal{O}_0 = \mathcal{C}^\infty(M_0)$,*
(2) *and satisfies that the cohomology of the degree $+1$ vector field $\delta\colon \mathcal{O}_- \to \mathcal{O}_-$ is given by*

$$H^i(\mathcal{O}_-, \delta) = \begin{cases} \mathcal{C}^\infty(M_0)/I, & i = 0 \\ 0, & i < 0 \end{cases} \tag{3.7}$$

Example 3.2. For $M_0 = \mathbb{R}^n$ and I the ideal of functions vanishing at 0, the graded algebra exterior form $\Omega(M_0)$ equipped with $\delta = \mathfrak{i}_E$ the contraction with the Euler vector field is a Koszul-Tate resolution of I. In that case, only $E_{+1} = TM_0$ is non-zero and the curvature is the Euler vector field.

We start with a few remarks that may help to understand the notion.

Remark 3.3. Item (1) in Definition 3.1 implies that the associated canonical graded vector bundle $E_\bullet$ of a Koszul-Tate resolution is concentrated in positive degrees $E_\bullet = \oplus_{i \geq 1} E_{+i}$. □

Remark 3.4. Let $\kappa \in \Gamma(E_{+1})$ be the curvature of a Koszul-Tate resolution. The condition on $H^0(\mathcal{O}, \delta)$ in Definition 3.1 implies that the curvature ideal $\langle \kappa \rangle$ of $\kappa \in \Gamma(E_{+1})$ coincides with I, i.e. a function $F \in \mathcal{C}^\infty(M_0)$ belongs to I if and only if there exists a section $\alpha \in \Gamma(E^*_{+1})$ such that $F = \langle \kappa, \alpha \rangle = \delta(\alpha)$. Moreover, for any Koszul-Tate resolution of an ideal I:

$$I + \mathcal{I}_- = \mathcal{O}\delta(\mathcal{I}_-) + \mathcal{I}_-,$$

where $\mathcal{I}_- = \oplus_{i \leq -1} \mathcal{O}_i$. □

We will need a variation of Definition 3.1.

[7] We use the standard notations from [12].

Definition 3.5. *Let $I \subset \mathcal{C}^\infty(M_0)$ be an ideal, and consider a positively-graded variety $\mathcal{K}_+$ on $\mathcal{C}^\infty(M_0)/I$. A* Koszul-Tate resolution of $\mathcal{K}_+$ *is a pair made of*

(1) *a splitting of $\mathcal{K}_+$, i.e.*

$$\mathcal{K}_+ \simeq \Gamma_I \left(S \left(\oplus_{i \geq 1} E^*_{-i} \right) \right),$$

(2) *a Koszul-Tate resolution of I with splitting*

$$\left(\Gamma \left(S \left(\oplus_{i \geq 1} E^*_i \right) \right), \delta \right),$$

assembled into a Q-manifold $(M_0, \mathcal{O}, \tilde{\delta})$ with splitting

$$\mathcal{O} \simeq \Gamma \left(\tilde{S} \left(\oplus_{i \neq 0} E^*_i \right) \right)$$

where $\tilde{\delta}$ is the extension of δ which is identically 0 on $\oplus_{i \geq 1} \Gamma \left(E^*_{-i} \right)$.

Here are a few comments about Definition 3.5. First, we need to point an important, but un-natural, effect of our conventions.

Warning 3.6. *Let $(M_0, \mathcal{O})$ be a graded manifold with splitting, recall that so-called* negative degree $\deg_-$ *is non-negative, and valued in $\mathbb{N}_0$. For a Koszul-Tate resolution $(M_0, \mathcal{O}_-, \delta)$ of an ideal i, the negative degree is simply the opposite of the degree: with respect to the negative degree, one now has $H^i(\mathcal{O}_-, \delta) = 0$ for $i \geq 1$ and $H^0(\mathcal{O}_-, \delta) = \mathcal{C}^\infty(M_0)$.*

One has to have this convention in mind to understand the following statement.

Lemma 3.7. *Let $(M_0, \mathcal{O}, \tilde{\delta})$ be a Koszul-Tate resolution of $\mathcal{K}_+$ as in Definition 3.5.*

(1) *Its zero locus positively graded Q-variety $\dfrac{\mathcal{O}}{\mathcal{O}\tilde{\delta}(\mathcal{I}_-) + \mathcal{I}_-}$ is $(\mathcal{K}_+, 0)$.*

(2) *The cohomology of the complex $(\mathcal{O}, \tilde{\delta})$ is given by:*

$$H^i(\mathcal{O}, \tilde{\delta}) = \begin{cases} \mathcal{K}_+, & i = 0 \\ 0, & i > 0 \end{cases} \tag{3.8}$$

Here the degree considered is the negative degree $\deg_-$.

Proof. The first item is a direct consequence of the identification:

$$\mathcal{O}\tilde{\delta}(\mathcal{I}_-) + \mathcal{I}_- = I\mathcal{O} + \mathcal{I}_- = \langle I + \Gamma \left(\oplus_{i \geq 1} E^*_i \right) \rangle.$$

Let us prove the second item. The cohomology of the complex

$$\left(\Gamma(S(\oplus_{i \geq 1} E^*_{-i})) \otimes_{\mathcal{C}^\infty(M_0)} \Gamma(S(\oplus_{i \geq 1} E^*_{+i})), \mathrm{id} \otimes \delta \right) \tag{3.9}$$

is $\Gamma(S \oplus_{i \geq 1} E^*_{-i}) \otimes_{\mathcal{C}^\infty(M_0)} \mathcal{C}^\infty(M_0)/I \simeq \mathcal{K}_+$. Now, $(\mathcal{O}, \tilde{\delta})$ is the completion of the complex (3.9) with respect to the negative degree, but completion does not affect cohomology, and the result follows. ■

We conclude the section with an important definition. To any Q-manifold $(M_0, \mathcal{O}, Q)$ was associated in Definition 1.17 another Q-manifold, called its negative part $(M_0, \mathcal{O}/\mathcal{I}_+, \delta)$.

Definition 3.8. *We say that a $\mathbb{Z}^*$-graded Q-manifold $(M_0, \mathcal{O}, Q)$ with curvature κ has a* Koszul-Tate negative part *if its negative part is a Koszul-Tate resolution of the curvature ideal $\langle \kappa \rangle$.*

3.2. **Vector fields on Koszul-Tate resolutions I: the cohomology.** The graded space $\mathfrak{X}(\mathcal{O}_-)$ of vector fields on a Koszul-Tate resolution $(M_0, \mathcal{O}_-, \delta)$ of an ideal $I \subset \mathcal{C}^\infty(M_0)$ is a DGLA when equipped with the graded commutator and the differential ad_δ. In particular, $(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta)$ is a complex, whose cohomology we now compute.
To start with, let us notice that $((\mathcal{O}_-\delta(\mathcal{I}_-)+\mathcal{I}_-)\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta)$ is a subcomplex of $(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta)$. By Remark 3.4, $\mathcal{O}_-\delta(\mathcal{I}_-) + \mathcal{I}_- = I + \mathcal{I}_-$, so that the quotient $\mathcal{O}_-/(\mathcal{O}_-\delta(\mathcal{I}_-) + \mathcal{I}_-)$ is isomorphic to $\mathcal{C}^\infty(M_0)/I$. This implies that:

$$\frac{\mathfrak{X}(\mathcal{O}_-)_i}{((\mathcal{O}_-\delta(\mathcal{I}_-) + \mathcal{I}_-)\mathfrak{X}(\mathcal{O}_-))_i} \simeq \begin{cases} 0 & i \leq -1 \\ \Gamma_I(TM_0) & i = 0 \\ \Gamma_I(E_i) & i \geq 1 \end{cases}$$

The quotient complex is therefore canonically isomorphic to a complex of the form

$$\Gamma_I(TM) \mapsto \Gamma_I(E_{+1}) \mapsto \Gamma_I(E_{+2}) \mapsto \cdots . \tag{3.10}$$

See Equation (3.6) for the notation Γ_I.

Definition 3.9. *Let $(M_0, \mathcal{O}_-, \delta)$ be a Koszul-Tate resolution of an ideal $I \subset \mathcal{C}^\infty(M_0)$. We call* linearization *of Koszul-Tate differential δ at the zero locus the complex* (3.10)*, and denote it by $(\mathfrak{X}_{lin}, \delta_{lin})$.*

Remark 3.10. As noticed in Section 1.1 in [4], the complex (3.10) can be understood as follows when I is the vanishing ideal of a subset $X_I \subset M_0$: first $\Gamma_I(E_{-i})$ or $\Gamma_I(TM_0)$ are the space of sections of E_{-i} or TM_0 on X_I. The differential of the curvature $\kappa : M_0 \to E_{+1}$ is a vector bundle morphism:

$$T\kappa \colon TM_0 \to TE_{+1}$$

over $\kappa : M_0 \to E_{+1}$ Now, for any $m \in X_I$, since $\kappa(m) = 0_m$, there is a canonical decomposition $T_{0_m}E_{+1} = T_mM_0 + E_{+1}|_m$, $pr_2 \circ T_m\kappa$ can be seen as a linear map $T_mM_0 \to E_{+1}|_m$, where pr_2 being the projection onto the second component. This map easily checked to coincide with the first bundle morphism in (3.10). All remaining morphisms in (3.10) are simply the restriction to X_I of the component of polynomial degree 0 of δ (which is by construction a degree +1 vector bundle endomorphism of $\oplus_{i\geq 1}E_i$). □

Although the context is not exactly the same, the linearization of the Koszul-Tate resolution matches the complex that appears in [1], Equation (4).
Let us now state the main result of this section. We start with some comment on degree and a Remark about degree 0 vector fields on a Koszul-Tate resolution.

From now on, we again use the negative grading. For a Koszul-Tate resolution (see Warning 3.6), it is simply the opposite of the degree. This has consequences when dealing with vector fields, whose degree are also changed by their opposite, and ad_δ is now a degree -1 operator.

Remark 3.11. A ad_δ-cocycle $q \in \mathfrak{X}(\mathcal{O}_-)$ of negative degree 0 induces a vector field $\underline{q} \in \mathfrak{X}(M_0)$ which satisfies $\underline{q}[I] \subset I$, and therefore induces a derivation q^I of $\mathcal{C}^\infty(M_0)/I$. If q is an ad_δ-coboundary, then $q^I = 0$. □

Theorem 3.12. *Let $(M_0, \mathcal{O}_-, \delta)$ be a Koszul-Tate resolution of an ideal $I \in \mathcal{C}^\infty(M_0)$. With respect to the negative degree, we have:*

$$H^{-i}(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta) = \begin{cases} H^{-i}(\mathfrak{X}_{lin}, \delta_{lin}), & i \leq 0 \\ 0, & i > 0. \end{cases} \tag{3.11}$$

Also, an ad_δ-cocycle q of degree 0 is a coboundary if and only if its induced derivation q^I of $\mathcal{C}^\infty(M_0)/I$ is zero.

Proof. Let us chose a splitting $\mathcal{O}_- \simeq \Gamma\left(S(\oplus_{i\geq 1} E_i^*)\right)$, and a family of affine connections ∇^k on E_k^* (recall Remark 1.11 for notations). Consider the following bigrading (on the "North-West" quarter):

$$\mathfrak{X}(\mathcal{O}_-)_{a,b} = \begin{cases} \mathcal{O}_a \otimes \mathfrak{X}(M_0) & \text{for } a \geq 0 \text{ and } b = 0, \\ \mathcal{O}_a \otimes \Gamma(E_{-b}) & \text{for } a \geq 0 \text{ and } b \leq -1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

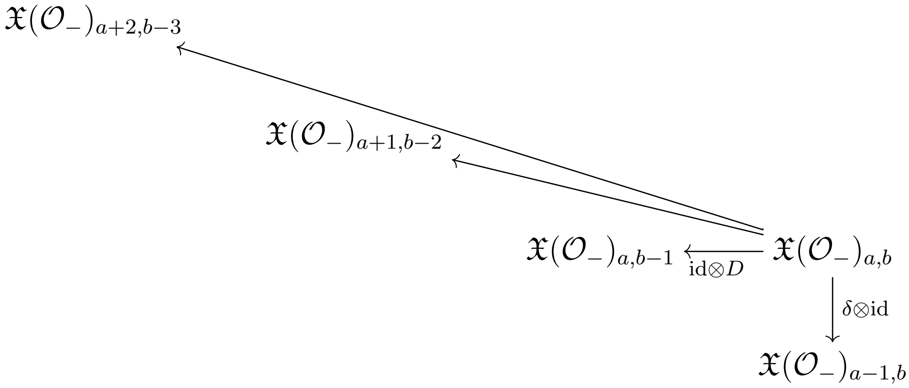
Here, $\mathcal{O}_a$ stands for functions of negative degree $+a$, i.e. of total degree $-a$. We adopt the following conventions:

1. All tensor products are over $\mathcal{C}^\infty(M_0)$.
2. A section $e \in \Gamma(E_b)$ is seen as the vertical vector field given by the derivation $\mathfrak{i}_e$ of $\mathcal{O}_-$. Notice that for the negative degree, it is of degree $-b$.
3. A vector field $X \in \mathfrak{X}(M_0)$ is extended to a degree 0 derivation of $\mathcal{O}_-$ by $X[\epsilon_k] = \nabla^k_X \epsilon_k$ for every section $\epsilon_k \in \Gamma(E_k^*)$.

It is indeed a bigrading, since:

$$\mathfrak{X}(\mathcal{O}_-)_i = \oplus_{a\geq 0} \mathfrak{X}(\mathcal{O}_-)_{a,i-a},$$

(infinite sums are allowed, since they converge with respect to the filtration $(F^i\mathcal{O})_{i\geq 0}$). With respect to this bi-grading, ad_δ decomposes as follows:



We can now use generic diagram chasing arguments: since all vertical lines are acyclic in degree $\neq 0$, the cohomology is concentrated in the 0-th cohomology of the line $a = 0$, which coincides with $\Gamma_I(TM_0)$ for $b = 0$ and $\Gamma_I(E_b)$ for $b \leq -1$. Equipped with the induced differential, a direct computation shows that it coincides (with opposite signs) with the differential of the complex (3.10). This proves (3.11).
Since all vertical lines are exact, a degree 0 ad_δ-cocycle is exact if and only if its bi-degree $(0, 0)$ component lies in the image of the vertical lines, i.e. belong to $I \otimes \mathfrak{X}(M_0)$.

Equivalently, this means that this element induces the zero map on $\mathcal{C}^\infty(M_0)/I$. This completes the proof. ■

Here is an immediate consequence of Theorem 3.12.

Corollary 3.13. *Let $(M_0, \mathcal{O}, \tilde{\delta})$ be a Koszul-Tate resolution of $\mathcal{K}_+$ as in Definition 3.5. With respect to the negative degree:*

$$H^{-i}(\mathfrak{X}(\mathcal{O}), \mathrm{ad}_{\tilde{\delta}}) = \begin{cases} \mathcal{K}_+ \otimes_{\mathcal{C}^\infty(M_0)/I} (H^{-i}(\mathfrak{X}_{lin}, \delta_{lin})), & i < 0 \\ \mathcal{K}_+ \otimes_{\mathcal{C}^\infty(M_0)/I} (\oplus_{i\geq 1}\Gamma_I(E_{-i}) \oplus H^0(\mathfrak{X}_{lin}, \delta_{lin})), & i = 0 \\ 0, & i > 0 \end{cases} \tag{3.13}$$

Proof. As in the proof of Theorem 3.12, one can use a family of connections on $(E_i)_{i\in\mathbb{Z}^*}$ to decompose the $\mathcal{O}$-module $\mathfrak{X}(\mathcal{O})$ as the sum of two submodules: one is

$$\mathcal{O} \otimes (\oplus_{i\geq 1} E_i \oplus TM_0) \text{ and } \mathcal{O} \otimes (\oplus_{i\geq 1} E_{-i})$$

Both modules are $\mathrm{ad}_{\tilde{\delta}}$-stable. On the second one, $\mathrm{ad}_{\tilde{\delta}} = \tilde{\delta} \otimes \mathrm{id}$, so that the cohomology is concentrated in negative degree 0 and coincides with

$$\frac{\Gamma(S(\oplus_{i\geq 1} E^*_{-i}))}{I} \otimes_{\mathcal{C}^\infty(M_0)} \Gamma(\oplus_{i\geq 1} E_{-i}) = \mathcal{K}_+ \otimes_{\mathcal{C}^\infty(M_0)/I} \Gamma_I(\oplus_{i\geq 1} E_{-i}).$$

The first one is the completion of the tensor product of $\Gamma(S(\oplus_{i\geq 1} E^*_{-i}))$ with the module $\mathfrak{X}(\mathcal{O}_-)$ of vector fields on a Koszul-Tate resolution of I, whose cohomology is given in Theorem 3.12. The differential being given by $\mathrm{id} \otimes \mathrm{ad}_{\tilde{\delta}}$, the result then follows from Theorem 3.12 and the fact that $\mathcal{K}_+$ is a $\mathcal{C}^\infty(M_0)/I$-projective module, so that tensoring with $\mathcal{K}_+$ preserves cohomology. ■

3.3. **Vector fields on Koszul-Tate resolutions II: the extension.** We now consider another problem. As stated in Remark 3.11, for a Koszul-Tate resolution $(M_0, \mathcal{O}_-, \delta)$ of an ideal I, an ad_δ-cocycle $q \in \mathfrak{X}(\mathcal{O}_-)$ induces a derivation q^I of $\mathcal{C}^\infty(M_0)/I$, and an ad_δ-coboundary induces a derivation equal to zero. In particular, there is a Lie algebra morphism

$$H^0(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta) \longrightarrow \mathrm{Der}(\mathcal{C}^\infty(M_0)/I). \tag{3.14}$$

The second part of Theorem 3.12 implies that this morphism is injective. The following statement shows that it is surjective.

Proposition 3.14. *Let $(M_0, \mathcal{O}_-, \delta)$ be a Koszul-Tate resolution of an ideal $I \subset \mathcal{C}^\infty(M_0)$. Then the natural Lie algebra morphism 3.14 is an isomorphism*

$$H^0(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta) \simeq \mathrm{Der}(\mathcal{C}^\infty(M_0)/I).$$

In particular, every derivation q^I of $\mathcal{C}^\infty(M_0)/I$ is induced by a degree 0 vector field $q \in \mathfrak{X}(\mathcal{O}_-)$ such that $[\delta, q] = 0$.

Proof. Denote the projection $\mathcal{C}^\infty(M_0) \to \mathcal{C}^\infty(M_0)/I$ by $F \mapsto \overline{F}$. Also, let us choose $(U_k, \chi_k)_{k\in K}$ a partition of unity of the manifold M_0 for which each U_k is a coordinate neighborhood on which each one of the vector bundles E_k admits a trivialization.

Let q^I be a derivation of $\mathcal{C}^\infty(M_0)/I$. Let $x_1, \dots, x_n$ be local coordinates on the open subset U_k for some $k \in K$. Consider any functions $F_1, \dots, F_r \in \mathcal{C}^\infty(U_k)$ such that $q^I(\overline{x_i}) = \overline{F_i}$. The vector field

$$\nu_k^0 := \sum_{i=1}^r F_i \frac{\partial}{\partial x_i}$$

satisfies by construction that $\nu_k^0(I) \subset I$, since it induces the derivation of $\mathcal{C}^\infty(U_k)/I$ which coincides with the restriction of q^I to U_k. These local vector fields ν_k^0 can be glued to a vector field ν^0 on M_0:

$$\nu^0 = \sum_{k \in K} \chi_k \nu_k^0.$$

This vector field still satisfies $\nu^0[I] \subset I$ by construction.
Since $I = \delta(\Gamma(E_{+1}^*))$, for any local trivialization $\eta_1, \dots, \eta_r$ of E_{+1}^*, defined on the open subset U_k, there exist functions $(\phi_{i,j})_{i,j=1}^r$ in $\mathcal{C}^\infty(U_k)$ such that the collection of functions $\kappa_i := \delta(\eta_i)$, $i = 1, \dots, r$ locally generates the vanishing ideal of the zero locus and

$$\nu^0 \delta(\eta_i) = \nu^0(\kappa_i) = \sum_{j=1}^r \phi_{i,j} \kappa_j = \sum_{j=1}^r \phi_{i,j} \delta(\eta_j)$$

Consider the vector field

$$\nu_k^1 := \sum_{i,j=1}^r \phi_{j,i} \eta_i \frac{\partial}{\partial \eta_j}.$$

By construction, it satisfies

$$\nu^0 \circ \delta = \delta \circ \nu_k^1.$$

Since δ is $\mathcal{C}^\infty(M_0)$-linear, the vector field

$$\nu^1 := \sum_{k \in K} \chi_k \nu_k^1$$

also satisfies:

$$\nu^0 \circ \delta = \delta \circ \nu^1.$$

Now, ν^0, ν^1 extends to vector fields on $\mathcal{O}_-$, that we will denote by the same symbol. The proof then consists in constructing recursively $\nu^j \in \bigoplus_{b \leq -j} \mathfrak{X}(\mathcal{O}_-)_{a,b}$ such that

$$\nu^j \circ \delta = \delta \circ \nu^{j+1}.$$

Assume that $\nu_0, \dots, \nu_j$ are constructed. Then notice that that $\nu^j \circ \delta$ is valued in the kernel of δ:

$$\delta \circ \nu^j \circ \delta = \nu^{j-1} \circ \delta^2 = 0.$$

Since the cohomology of $(\mathcal{O}_-, \delta)$ is zero, $\nu^j \circ \delta$ is therefore valued in the image of δ, and since $\mathcal{O}_-$ is a projective $\mathcal{C}^\infty(M_0)$-module, the existence of the vector field ν^{j+1} is granted. Moreover, with respect to the bi-grading above (see Equation (3.12)),
$\nu^{j+1} \in \bigoplus_{b \leq -(j+1)} \mathfrak{X}(\mathcal{O}_-)_{a,b}$. As a consequence, the sequence

$$q^k := \sum_{j=0}^k \nu^j, \tag{3.15}$$

converges and the limit is the desired vector field q. ■

Consider now a Koszul-Tate resolution of a graded variety given by $\mathcal{K}_+$ as in Definition 3.5. Again, notice that a total degree k and negative degree 0 vector field q_0 such that $[\tilde{\delta}, q_0] = 0$ induces a degree k derivation of $\mathcal{K}_+$. If q_0 is an $\mathrm{ad}_{\tilde{\delta}}$-cocycle, that derivation is zero. Proposition 3.14 extends easily to give the following result.

Corollary 3.15. *Let $(M_0, \mathcal{O}, \tilde{\delta})$ be a Koszul-Tate resolution of a positively graded variety $\mathcal{K}_+$ as in Definition 3.5. There is a natural isomorphism:*

$$H^{(0,k)}\left(\mathfrak{X}(\mathcal{O}), \mathrm{ad}_{\tilde{\delta}}\right) = \mathrm{Der}^k(\mathcal{K}_+)$$

where $H^{(0,k)}$ stands for the cohomology in negative degree 0 and total degree k and Der^k stands for derivations of degree k of $\mathcal{K}_+$. In particular, for any degree k derivation Q_+ of $\mathcal{K}_+$, there exists $q_0 \in \mathfrak{X}(\mathcal{O})$ of negative degree 0 and total degree k satisfying $[\tilde{\delta}, q_0] = 0$ and inducing the derivation Q_+ on $\mathcal{K}_+$.

Remark 3.16 (Extension of derivations in the affine case)**.** At the beginning of the proof of Proposition 3.14, it was shown that in the smooth category any derivation of the quotient algebra $\mathcal{C}^\infty(M_0)/I$, considered as functions on the zero locus, can be extended to a derivation of the entire algebra of functions $\mathcal{C}^\infty(M_0)$. This is also true in algebraic geometry for functions on an affine variety.
Let M_0 be an affine $n-$dimensional space over a field $\Bbbk$ of characteristic 0 (we think of it as $\mathbb{R}$ or $\mathbb{C}$) with affine coordinates $(z^i)_{i=1}^n$; and $I \subset \Bbbk[z^1, \ldots, z^n]$ be an ideal, then every derivation of $\mathcal{K} = \Bbbk[z^1, \ldots, z^n]/I$ admits an extension to a derivation of $\Bbbk[z^1, \ldots, z^n]$. Indeed, let q^I be a derivation of $\mathcal{K}$. Define q such that $q\big(z^i\big)$ equals to the preimage of $q^I[z^i] \in \mathcal{K}$ under the projection map $\Bbbk[z^1, \ldots, z^n] \to \mathcal{K}$, where $[z^i] = z^i/I$. The derivation q extends to the whole algebra of functions $\Bbbk[z^1, \ldots, z^n]$ by the Leibniz rule. It is easy to see that $q(I) \subset I$ and that q induces q^I. Checking the rest of the proof line by line, one sees that Proposition 3.14 remains valid in the context of affine varieties in algebraic geometry. □

Remark 3.17. The above statements (Theorem 3.12 and Proposition 3.14) can be proved in a the following alternative way. Let $\mathfrak{X}_{\text{null}} = \mathfrak{X}_{\text{null}}(\mathcal{O}_-)$ be the graded Lie subalgebra of all derivations v of $\mathcal{O}_-$ satisfying $v(\mathcal{O}_-) \subset I + \mathcal{I}_-$. We have a short exact sequence of complexes

$$0 \to \big(\mathfrak{X}_{\text{null}}, \mathrm{ad}_\delta\big) \to \big(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta\big) \to \big(\mathfrak{X}_I, \mathrm{ad}_\delta\big) \to 0$$

and it is easy to check that $\big(\mathfrak{X}_I, \mathrm{ad}_\delta\big)$ coincides with (3.10). This short exact sequece leads to the long exact sequence in cohomology

$$\ldots \to H^i\big(\mathfrak{X}_{\text{null}}, \mathrm{ad}_\delta\big) \to H^i\big(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta\big) \to H^i\big(\mathfrak{X}_I, \mathrm{ad}_\delta\big) \to H^{i+1}\big(\mathfrak{X}_{\text{null}}, \mathrm{ad}_\delta\big) \to \ldots$$

It is relatively easy to check that the complex $\big(\mathfrak{X}_{\text{null}}, \mathrm{ad}_\delta\big)$ is acyclic. Combining this statement with the fact that any derivation of $\mathcal{C}^\infty(M_0)/I$ extends to a derivation of $\mathcal{O}_-$ (see the beginning of the Proposition 3.14), we prove Theorem 3.12 and the remaining part of Proposition 3.14 altogether. □

Remark 3.18.

- In fact, Theorem 3.12 computed the cohomology of $(\mathfrak{X}(\mathcal{O}_-), \delta)$ by using the spectral sequence associated to the following filtration of $\mathfrak{X}(\mathcal{O}_-)$: $F^p\mathfrak{X}(\mathcal{O}_-)$ is the Lie subalgebra of vector fields on M which annihilate the subalgebra of functions generated by all elements of negative degree $0, \dots, p-1$. This spectral sequence converges in the second term. Its zero term gives sections of the graded vector bundle $\pi_-^*\big(TM_0 \oplus E_+\big)$, where $E_+ = \bigoplus_{k>0} E_k$ and π_- is the projection of $(M_0, \mathcal{O})$ onto M_0, while the first term of the spectral sequence – sections of the restriction of $TM_0 \oplus E_+$ on X together with the differential determined by the normal linearization of δ along X.

- The "restriction" of vector fields to the zero locus $\mathfrak{X}_I(\mathcal{O})$ is an $\mathbb{N}-$ graded $\mathcal{C}^\infty(M_0)/I$-module, the homogeneous components of which of negative degree $i = 0$ and $i < 0$ are canonically isomorphic to $\mathfrak{X}_I(\mathcal{O}_-)$ and $\mathcal{K} \otimes_{\mathcal{O}_-} \Gamma(E_{-i})$, respectively.
- $\oplus_i H^i(\mathfrak{X}_{lin}, \delta_{lin})$ is a graded Lie-Rinehart $\mathcal{C}^\infty(M_0)/I$-algebra, which extends $\mathfrak{X}_I(\mathcal{O}_-)$; furthermore, the latter is embedded into the former as a Lie-Rinehart $\mathcal{C}^\infty(M_0)/I$-subalgebra consisting of all elements of negative degree.

□

It follows from Theorem 3.12 that the ad_δ-cohomology of vector fields on a Koszul-Tate resolution $(M_0, \mathcal{O}_-, \delta)$ of I are zero near any point outside the zero locus of I. We also claim that non-trivial cohomologies of non-zero degree only appear on the singular part of the zero locus of I.Example 3.19 illustrate this phenomenon and shows that the positive degree part in cohomology of $\big(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta\big)$, where $\big(\mathcal{O}_0, \delta\big)$ is the Koszul-Tate resolution of an ideal $I \subset \mathcal{O}(M_0)$, is related to singularities of the zero locus of this ideal. In complement, Example 3.20 tells us that, even in the complete intersection case, the degree 1 cohomology of vector fields on a Koszul-Tate resolution can be non-trivial.

Example 3.19. Assume that $X \subset M_0$ is a smooth submanifold, I is the ideal of functions vanishing on X, δ is a Koszul-Tate differential which resolves I, such that X is the zero locus of δ regarded as a homological vector field on a non-positively graded M. It is possible to cover M by graded coordinate charts such that either such a chart does not intersect X, then the corresponding $\mathfrak{i}_\kappa$ is non-vanishing at all points, so we can use Lemma 2.5 and technique from Corollary 2.8 to show that the $\mathrm{ad}_\delta-$ cohomology of vector fields over this chart are vanishing, or there are adapted coordinates $(x^i, y^a, \eta^a, \xi^\alpha, \zeta^a)$ such that[8]

$$\delta = \sum_a y^a \frac{\partial}{\partial \eta^a} + \sum_\alpha \zeta^\alpha \frac{\partial}{\partial \xi^\alpha}\,.$$

In such case the intersection of the above coordinate chart with X is given by equations $y^a = \eta^a = \xi^\alpha = \zeta^\alpha = 0$, therefore all sections of the restriction of TM onto X are of the form

$$v(x, \frac{\partial}{\partial x}) + \sum_a \left(f^a(x) \frac{\partial}{\partial y^a} + h^a(x) \frac{\partial}{\partial \eta^a} \right) + \sum_\alpha \left(\lambda^a(x) \frac{\partial}{\partial \zeta^a} + \mu^\alpha(x) \frac{\partial}{\partial \xi^\alpha} \right)$$

It is easy to see that $\left\{ \frac{\partial}{\partial y^a}, \frac{\partial}{\partial \eta^a}, \frac{\partial}{\partial \zeta^a}, \frac{\partial}{\partial \xi^\alpha} \right\}$ generate an acyclic complex w.r.t. δ, therefore the cohomology of positive degree sections of $TM_{|X}$ are zero over this coordinate chart and thus on the whole X as δ is linear under the multiplication on functions on M_0 which allows us to apply the partition of unity technique.

[8]In mathematical physics $(y^a, \eta^a, \xi^\alpha, \zeta^a)$ would be called contractible pairs.

Example 3.20. On $M_0 = \mathbb{R}^2$, equipped with affine coordinates (x, y), let I be the ideal generated by xy, and X be an affine variety given by the equation $xy = 0$. A Koszul-tate resolution of I is determined by a homological vector field $\delta = xy\frac{\partial}{\partial \xi}$ on the non-positively graded affine manifold $(M_0, \mathcal{O}_-)$ with graded coordinates (x, y, ξ), where ξ has total degree -1. It is routine to check directly that, as stated in Proposition 3.14:

$$H^0(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta) \simeq \left\{ xf(x)\frac{\partial}{\partial x} + yg(y)\frac{\partial}{\partial y} \,\middle|\, f, g \in \mathcal{C}^\infty(\mathbb{R}) \right\}.$$

Since a degree 1 vector field is a ad_δ-cocycle, since $[\delta, \frac{\partial}{\partial x}] = y\frac{\partial}{\partial \xi}$, $[\delta, \frac{\partial}{\partial y}] = x\frac{\partial}{\partial \xi}$, and $[\delta, \xi\frac{\partial}{\partial \xi}] = xy\frac{\partial}{\partial \xi}$, and since the quotient of $\mathcal{C}^\infty(M_0)$ by the ideal generated by x, y, xy is $\mathbb{R}$, we also have

$$H^1(\mathfrak{X}(\mathcal{O}_-), \mathrm{ad}_\delta) = \left\{ \lambda\frac{\partial}{\partial \xi} \,\middle|\, \lambda \in \mathbb{R} \right\}.$$

In particular, the degree 1 cohomology is different from zero.

3.4. Koszul-Tate resolutions and singular locus positively-graded Q-variety.

Here is the main result of this section.

Theorem 3.21. *Let M_0 be a manifold and $I \subset \mathcal{C}^\infty(M_0)$ an ideal. Given*

(1) *a Koszul-Tate resolution $(M_0, \mathcal{O}_-, \delta)$ of I with a splitting*
$$\mathcal{O}_- = \Gamma(S(\oplus_{i\geq 1} E_i^*))$$
(2) *a positively graded Q-variety $(\mathcal{K}_+, Q_+)$ on $\mathcal{C}^\infty(M_0)/I$ with a splitting*
$$\mathcal{K}_+ = \Gamma_I(S(\oplus_{i\geq 1} E_{-i}^*)),$$

there exists a Q-manifold $(M_0, \mathcal{O}, Q)$ with a splitting:
$$\mathcal{O} \simeq \Gamma(\tilde{S}(\oplus_{i\in\mathbb{Z}^*} E_i^*))$$

(1) *whose negative part is the Koszul-Tate resolution $(M_0, \mathcal{O}_-, \delta)$,*
(2) *and whose zero locus positively graded Q-variety is $(\mathcal{K}_+, Q_+)$.*

Two such Q-vector fields are diffeomorphic through a diffeomorphism which is the composition of flows of degree zero vector fields as in Proposition 1.24, and that induce the identity maps of the base manifold M_0, of the negative part $\mathcal{O}_-$, and of the positively graded Q-variety $\mathcal{K}_+$.

Proof. The idea of the proof consists in applying perturbation theory techniques, and construct Q (and the degree 0 vector fields defining Ψ) through a recursion by showing that the obstructions for the next step are cohomology classes that vanish.
Let $(M_0, \mathcal{O}, \tilde{\delta})$ be as in Definition 3.5. The first step consists in applying Corollary 3.15: there exists a vector field q_0 of negative degree 0 and total degree $+1$, such that $[q_0, \tilde{\delta}] = 0$ (which implies that Q_0 induces a derivation of $\mathcal{K}_+$) whose induced derivation of $\mathcal{K}_+$ is Q_+.
Now, the proof of the existence of the vector field Q consists in constructing recursively a family $(q_i)_{i\geq 1}$ such that

(1) Each q_i is of negative degree i and total degree $+1$
(2) $Q_i = \tilde{\delta} + q_0 + \cdots + q_i$ satisfies $[Q_i, Q_i] = 0$ up to vector fields of negative degree $\geq i$.

For instance $Q_0 = \tilde{\delta} + q_0$ satisfies the recursion for $i = 0$ since:

$$[\tilde{\delta} + Q_0, \tilde{\delta} + Q_0] = [Q_0, Q_0]$$

and since $[Q_0, Q_0]$ is of negative degree 0.
Now, by the graded Jacobi identity of the graded Lie algebra of vector fields $\mathfrak{X}(\mathcal{O})$,

$$\left[\tilde{\delta} + Q_0, [\tilde{\delta} + Q_0, \tilde{\delta} + Q_0]\right] = 0,$$

so that $\mathrm{ad}_{\tilde{\delta}}([Q_0, Q_0]) = 0$ is an $\mathrm{ad}_{\tilde{\delta}}$-cocycle of negative degree 0. Now, since Q_0 induces Q_+ on $\mathcal{K}_+$ and since $Q_+^2 = 0$, the class of $[Q_0, Q_0]$ in $\mathcal{K}_+ \otimes H^0(\mathfrak{X}_{lin}, \delta_{lin})$ is zero, so that there exists a vector field q_1 of total degree $+1$ and negative degree $+1$ such that $[\tilde{\delta}, q_1] = [q_0, q_0]$. As a consequence

$$Q_1 = \tilde{\delta} + q_0 + q_1$$

satisfies the recursion condition for $i = 1$.
The proof then continues easily by noticing that if $Q_i := \tilde{\delta} + \sum_{k=1}^{i} q_k$ satisfies the recursion assumption at order i, then

$$[Q_i, Q_i] = \sum_{k=0}^{i} [q_k, q_{i-k}] + R_i$$

with R_i of negative degree $\geq i + 2$ and $\sum_{k=0}^{i} [q_k, q_{i-k}]$ being the component of negative degree $i + 1$. By the graded Jacobi identity, this implies that $\sum_{k=0}^{i} [q_k, q_{i-k}]$ is an $\mathrm{ad}_{\tilde{\delta}}$-cocycle of negative degree $i + 1$. Since cohomology is zero in that degree by Corollary 3.13, there exists a vector field q_{i+1} of total degree 1 and negative degree $i + 1$ such that $-\sum_{k=0}^{i} [q_k, q_{i-k}] = \mathrm{ad}_{\tilde{\delta}} q_{i+1}$ which in turn implies that

$$Q_{i+1} = \tilde{\delta} + \sum_{k=0}^{i+1} q_k$$

satisfies the recursion relation for $i + 1$. Now, the series

$$\tilde{\delta} + \sum_{i=0}^{\infty} q_i$$

converges with respect to the negative degree filtration $(F^i \mathcal{O})_{i \geq 0}$. We denote by Q its limit. By construction, $[Q, Q] = 0$ (since $[Q, Q]$ is a derivation that takes values in $\cap_{i \geq 0} F^i \mathcal{O} = \{0\}$), and Q has total degree $+1$, so that $(M_0, \mathcal{O}, Q)$ is a $\mathbb{Z}^*$-graded Q-manifold. By Remark 2.13, Q satisfies both requirements in the Theorem 3.21.

Now, let us show that any two such vector fields can be intertwined by a diffeomorphism for the desired form. Let Q and Q' be two vector fields as in Theorem 3.21. We will construct a family $u_1, u_2, u_3, \ldots$ of total degree 0 and of respective negative degrees $1, 2, 3, \ldots$ such that the sequence of degree $+1$ vector fields defined by the recursion relation $Q_0 = Q$ and

$$Q_{i+1} = e^{\mathrm{ad}_{u_{i+1}}} Q_i \tag{3.16}$$

(which is well-defined, see Section 1.5) satisfies that Q_i coincides with Q' in negative degrees $-1, \ldots, i - 1$. Proposition 1.24 implies then that the infinite composition of the

exponentials of the vector fields u_i intertwines Q and Q' through a diffeomorphism Ψ which is by construction of the desired form.
Let us first construct u_1. We have:

$$\begin{aligned} Q &= \tilde{\delta} + q_0 + \sum_{i\geq 1} q_i \\ Q' &= \tilde{\delta} + q'_0 + \sum_{i\geq 1} q'_i \end{aligned}$$

where q_i, q'_i are of negative degree i. Now, since both q_0, q'_0 are $\mathrm{ad}_{\tilde{\delta}}$-cocycles, so is $q_0 - q'_0$. Since by construction, both q_0 and q'_0 induce the same derivation Q_+ on $\mathcal{K}_+$, their difference induce the trivial derivation of $\mathcal{K}_+$. This implies that $q_0 - q'_0$ is a $\mathrm{ad}_{\tilde{\delta}}$-coboundary by Corollary 3.13, and there exists a vector field u_{-1} of negative degree $+1$ and total degree 0 such that

$$q_0 - q'_0 = [\tilde{\delta}, u_{-1}].$$

By construction,

$$Q_1 := e^{\mathrm{ad}_{u_{-1}}}(Q) = Q + \sum_{m=1}^{\infty} \frac{1}{m!} ad^m_{u_{-1}}(Q)$$

is well-defined, squares to zero, and satisfies again the requirements of Theorem 3.21. Also, it coincides with Q' in negative degree -1 and 0.
Now, assume that $u_1, \ldots, u_i$ are constructed. Consider the decompositions according to negative degrees:

$$\begin{aligned} Q' &= \tilde{\delta} + q'_0 + \cdots + q'_i + q'_{i+1} + \cdots \\ Q_i &= \tilde{\delta} + q'_0 + \cdots + q'_i + q_{i+1} + \cdots \end{aligned}$$

It follows from $[Q_i, Q_i] = 0$ and $[Q', Q'] = 0$ that

$$\mathrm{ad}_{\tilde{\delta}} q_{i+1} = -\sum_{k=0}^{i+1} [q_k, q_{i+1-k}] \text{ and } \mathrm{ad}_{\tilde{\delta}} q'_{i+1} = -\sum_{k=0}^{i+1} [q_k, q_{i+1-k}]$$

The difference $q_{i+1} - q'_{i+1}$ is therefore an $\mathrm{ad}_{\tilde{\delta}}$-cocycle. Since by Corollary 3.13, the cohomology is zero in degree $i+1$, there exists a vector field u_{i+1} (of negative degree $i+1$ and total degree 0) such that $q'_{i+1} = q_{i+1} + \mathrm{ad}_{\tilde{\delta}} u_{i+1}$. The vector field Q_{i+1} defined as in (3.16) satisfies the recursion relation for $i+1$.
By Proposition 1.24, the infinite ordered product of automorphisms $\Psi = \lim_{k\to\infty} e^{u_{-k}} \circ \ldots \circ e^{u_{-1}}$ exists and induces a diffeomorphism Ψ of the graded manifold M. Furthermore, one has $\Psi Q \Psi^{-1} = \lim_{k\to\infty} e^{\mathrm{ad}_{u_{-k}}} \circ \ldots \circ e^{\mathrm{ad}_{u_{-1}}}(Q)$ and

$$\Psi Q \Psi^{-1} - Q' \in \bigcap_{j\geq 0} F^j \mathcal{O}(M) = \{0\},$$

therefore $\Psi Q \Psi^{-1} - Q' = 0$.
It is also clear that, since the total degree of each u_i is zero, $deg_+(u_{-i}) = deg_-(u_{-i}) = i$ for each $i \geq 1$, so that the positive degree of $u_1, u_2, u_3, \ldots$ is $1, 2, 3, \ldots$ respectively. This implies that $\Psi(F) - F \in \mathcal{I}_+$ for every $F \in \mathcal{O}$, and therefore that Ψ induces the identity

on the negative part $\mathcal{O}_- = \mathcal{O}/\mathcal{I}_+$. These degree relations also imply that $\Psi(F) - F \in \mathcal{I}_-$, so that Ψ induces the identity of $S(\oplus_{i\geq 1}\Gamma(E^*_{-i}))$, and therefore of its quotient $\mathcal{K}_+$. ■
Here is an immediate consequence of the second part of Theorem 3.21.

Corollary 3.22. *Any two $\mathbb{Z}^*$-graded manifolds $(M_0, \mathcal{O}, Q)$ and $(M_0, \mathcal{O}, Q')$ over the same graded manifold $(M_0, \mathcal{O})$ whose negative parts coincide and are Koszul-Tate resolutions, and whose positively graded Q-varieties coincide, are diffeomorphic through a diffeomorphism as in Theorem 3.21.*

Theorem 3.21 has several geometric consequences. Let $W \subset M$ be a subset such that the ideal I_W of functions vanishing on it admits a Koszul-Tate resolution $(\mathcal{O}_-, \delta)$. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A linear map:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{X}(M) \\ x & \mapsto & \underline{x} \end{array}$$

is said to be an *action off-shell closing on-shell* if its satisfies

$$\underline{x}[I_W] \subset I_W \text{ and } [\underline{x}, \underline{y}] - \underline{[x, y]} \in I_W \mathfrak{X}(M).$$

The first condition means that the vector fields composing the infinitesimal action of $\mathfrak{g}$ are tangent to W, and the second one means that the restriction to W of $x \mapsto \underline{x}$ is a Lie algebra morphism. Such an action off-shell closing on-shell can be seen as a positively graded Q-variety on $\mathcal{C}^\infty(M)/I_W$ whose splitting is given by $E_{-1} = \mathfrak{g} \times M \to M$ and $E_{-i} = 0$ for $i \geq 1$ with vector field $Q_+ = d^{CE} \oplus Y_0$ with $Y_0 \in \mathfrak{g}^* \otimes \mathfrak{X}(M)$. Here d^{CE} is the Chevalley-Eilenberg differential, and Y_0 represents the map $x \mapsto \underline{x}$.

Corollary 3.23. *For any action off-shell closing on-shell of a finite dimensional Lie algebra $\mathfrak{g}$ on a subset W admitting a Koszul-Tate resolution $(\mathcal{O}_-, \delta)$, there exists a L_∞-algebra morphism from $\mathfrak{g}$ to the DGLA of vector field on $(\mathcal{O}_-, \delta)$.*

Proof. Theorem 3.21 states that there is a vector field Q on a $\mathbb{Z}^*$-graded manifold with the ring of functions $\wedge^\bullet \mathfrak{g}^* \otimes \mathcal{O}_-$. A closer look at the construction of the vector field Q shows that it can be chosen to be of the form:

$$Q = \delta + d^{CE} + X_0 + X_1 + X_2 + \cdots$$

with d^{CE} being the Chevalley-Eilenberg operator of $\mathfrak{g}$, and with $X_i \in \wedge^i \mathfrak{g}^* \otimes \mathfrak{X}(\mathcal{K})_{-i+1}$. Each X_i can be seen as a map from $\wedge^i \mathfrak{g}$ to $\mathfrak{X}(\mathcal{K})_{-i})$. It is routine to check that $Q^2 = 0$ implies that the sequence $\{X_i\}_{i\geq 0}$ is the sequence of Taylor coefficients of the required L_∞-morphism. □

Example 3.24. Corollary 3.23 implies that any vector field tangent to W can be extended to a vector field on any of its Koszul-Tate resolution $(\mathcal{K}, \delta)$ commuting with δ.

Example 3.25. Consider an Hamiltonian actions of a Lie algebra $\mathfrak{g}$ on a Poisson manifold (M, π) with momentum map μ. If the components $\sum_i \mu_i \epsilon_i$ of μ are independent functions, then a Koszul-Tate resolution of the ideal of functions vanishing on $\{\mu = 0\}$ is given by the graded manifold whose functions of degree $-i$ are $C^\infty(M, \wedge^i \mathfrak{g})$, equipped with the vector field $\mathfrak{i}_\mu$. This happens in particular when μ is a submersion. The construction of Corollary 3.23 then applies and gives back, when appropriate choices are made, a Q-manifold which we now describe. As a graded manifold, it is given by $E_{-1} = \mathfrak{g} \times M \to M$,

$E_{+1} = \mathfrak{g}^* \times M$ and $E_i = 0$ otherwise. The vector field is given by

$$Q = \sum_{i=1}^{r} \mu_i \frac{\partial}{\partial \eta_i} + \sum_{i=1}^{r} \theta^i \left(X_{\mu_i} - \sum_{j,k=1}^{r} C_{ij}^k \eta_k \frac{\partial}{\partial \eta_j} \right) - \frac{1}{2} \sum_{i,j,k=1}^{r} C_{ij}^k \theta^i \theta^j \frac{\partial}{\partial \theta^k}$$

where (e_i) is a basis of $\mathfrak{g}$, and θ^i, η_i are the corresponding variables of respective degree $+1$ and -1, μ_i the corresponding components of the momentum mapping and X_{μ_i} is its Hamiltonian vector field. Also, C_{ij}^k stand for the Christoffel symbols of $\mathfrak{g}$.

4. Local structures near points where the curvature vanishes

4.1. Q-structure in local coordinates. In the following, we consider local coordinates of a graded manifold $(M_0, \mathcal{O})$ of the form

$$(y_1, \ldots, y_r, (x_i)_{i \in I}, \theta_1, \ldots, \theta_r, (\eta_j)_{j \in J}),$$

where Latin letters x_i, y_k will be used for degree 0 variables, the Greek letters θ_k (resp. η_j) will be used for variables of degree $+1$ (resp. of degree different from 0). Last, we will also assume that the variables y_k and θ_k go "in pairs" and that there is the same number r of them. Last, an expression like

$$R\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right)$$

stand for any local vector field of the form:

$$\sum_{i \in I} A_i(x,\eta) \frac{\partial}{\partial x_i} + \sum_{j \in J} B_j(x,\eta) \frac{\partial}{\partial \eta_j}$$

where $A_i(x,\eta), B_j(x,\eta)$ are functions that depend on the variables $(x_i)_{i\in I}, (\eta_j)_{j \in J}$ only. For any Q-manifold $(M_0, \mathcal{O}, Q)$, equipped with a splitting

$$\Phi \colon \mathcal{O} \simeq \Gamma\left(\hat{S} \oplus_{i \in \mathbb{Z}^*} E_i^*\right),$$

the anchor map $\rho \colon E_{-1} \to TM_0$ is the vector bundle morphism defined by:

$$\langle Q[f]^{(1)}, u \rangle = \rho(u)[f],$$

for every $u \in \Gamma(E_{-1})$ and $f \in \mathcal{C}^\infty(M_0)$. Above, $Q[f]^{(1)}$ stands for the component of polynomial degree 1 of $Q[f]$: since $Q[f]$ is of degree 1, $Q[f]^{(1)}$ is a section of E_{-1}^*, so that the previous definition makes sense.

Remark 4.1. By construction, the anchor map of a Q-manifold is a vector bundle morphism $\rho \colon E_{-1} \to TM_0$ that depends on the choice of the splitting, although the vector bundles E_{-1} and TM_0 do not. For instance, in a splitting as in Proposition 2.3 for which $Q = \mathfrak{i}_\kappa$, the anchor map is the zero map. But it may be non-zero in some other splitting. However, at every m that belongs to the zero locus of the curvature $\kappa \in \Gamma(E_{+1})$, the anchor map $\rho \colon E_{-1} \to M$ does not depend on the choice of a splitting, and is therefore canonical. □

Remark 4.1 implies that the following theorem only makes sense when the point m is the zero locus of the curvature κ.

Theorem 4.2. *Let $(M_0, \mathcal{O}, Q)$ be a Q-manifold. Let $\rho\colon E_{-1} \to TM_0$ be the anchor map corresponding to some splitting. Every point $m \in M_0$ on the zero locus of the curvature κ admits a coordinate neighborhood with variables $(y_1, \dots, y_r, (x_i)_{i\in I}, \theta_1, \dots, \theta_r, (\eta_j)_{j\in J})$ on which Q reads:*

$$Q = \sum_{k=1}^{r} \theta_k \frac{\partial}{\partial y_k} + R\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right)$$

where r is the rank of the anchor map $\rho\colon E_{-1} \to TM_0$ at m.

We start with a remark and two lemmas, before proving a proposition crucial for the proof of the above theorem.

Remark 4.3. Any degree 0 vector field $\boldsymbol{v}$ on a $\mathbb{Z}^*$-graded manifold $(M_0, \mathcal{O})$ induces a vector field $\underline{\boldsymbol{v}}$ on M_0: A degree 0 vector field being, by definition, a degree 0 derivation of $\mathcal{O}$, it preserves both negative and positive functions, so it preserves the maximal ideal $\mathcal{I}$, and induces a derivation of the quotient $\mathcal{O}/\mathcal{I}$, which is isomorphic to $\mathcal{C}^\infty(M_0)$. This induced derivation is a vector field on M_0. In coordinates, this assignment reads:

$$\begin{array}{rcl} \mathfrak{X}(\mathcal{O})_0 & \to & \mathfrak{X}(M_0) \\ \sum_i f_i(z,\zeta)\frac{\partial}{\partial z_i} + \sum_j g_j(z,\zeta)\frac{\partial}{\partial \zeta_j} & \mapsto & \sum_i f_i(z,0)\frac{\partial}{\partial z_i} \end{array}$$

where (z, ζ) are local coordinates of degree 0 and different from 0 respectively. □

Lemma 4.4 extends to graded manifolds the well-known straightening theorem, also known as Hadamard Lemma.

Lemma 4.4. *Let $\boldsymbol{v}$ be a vector field of degree 0 on a graded manifold $(M_0, \mathcal{O})$ with M_0. Every point of the base manifold M_0 where the induced vector field $\underline{\boldsymbol{v}}$ is different from zero admits a coordinate neighborhood $(y, (x_i)_{i\in I}, (\eta_j)_{j\in J})$ on which $\boldsymbol{v} = \frac{\partial}{\partial y}$.*

Proof. The proof is rather straightforward: use the general form of the coordinate changes on graded manifolds (cf. [20]). ■

The following lemma is the result of an obvious computation.

Lemma 4.5. *Every vector field Q, defined on a coordinate neighborhood $(y, x_\bullet, \eta_\bullet)$, that satisfies $\left[Q, \frac{\partial}{\partial y}\right] = 0$ is of the form:*

$$Q = \tau(x_\bullet, \eta_\bullet)\frac{\partial}{\partial y} + R\left(x_\bullet, \eta_\bullet, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right).$$

We can now prove the following statement:

Proposition 4.6. *Let $(M_0, \mathcal{O}, Q)$ be a Q-manifold equipped with a splitting. Let $\rho\colon E_{-1} \to TM_0$ be the corresponding anchor map. Every point $m \in M_0$ in the zero locus of the curvature κ such that $\rho_m : E_{-1} \to TM_0$ is not the zero map admits a coordinate neighborhood with variables $(y, (x_i)_{i\in I}, \theta, (\eta_j)_{j\in J})$ on which Q reads:*

$$Q = \theta\frac{\partial}{\partial y} + R\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right).$$

Proof. The map $\rho : E_{-1} \to TM_0$ is different from zero at $m \in M_0$ if and only if there exists a section e in $\Gamma(E_{-1})$ such that the degree 0 vector field $\boldsymbol{v} := [Q, \mathfrak{i}_e]$ (which is of degree 0) has a basic vector field $\underline{v}$ (see Remark 4.3) different from 0 at m. By Lemma 4.4, there exists a coordinate neighborhood $(y, x_\bullet, \eta_\bullet)$ on which $v = [Q, \mathfrak{i}_e] = \frac{\partial}{\partial y}$. Since $[v, Q] = 0$, Lemma 4.5 implies that in these coordinates:

$$Q = \tau(x,\eta)\frac{\partial}{\partial y} + R\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right).$$

Now, $\tau(x, \eta) = Q(y)$ is a degree $+1$ function whose component in $\Gamma((E_{-1})^*)$ cannot be zero in view of

$$\mathfrak{i}_e\tau(x,\eta) = \mathfrak{i}_e Q(y) = [\mathfrak{i}_e, Q](y) + Q(\mathfrak{i}_e[y]) = \frac{\partial}{\partial y}(y) + Q(\mathfrak{i}_e[y]) = 1 + Q(\mathfrak{i}_e[y])$$

and the fact that the projection of the degree 0 function $Q(\mathfrak{i}_e[y])$ on $\mathcal{C}^\infty(M_0)$ has to be an element of the zero locus ideal for degree reasons. We can therefore replace one of the degree -1 variables in the coordinates $\eta_\bullet$ by $\tau(x, \eta)$: we denote by θ this new variable. Since $\theta = \tau(x, \eta)$ does not depend on the variable y, this change of coordinates does not affect $\theta\frac{\partial}{\partial y}$ and changes R in a vector field that again does not depend on y nor contains $\frac{\partial}{\partial y}$. But it may contain a component in $\frac{\partial}{\partial \theta}$. In conclusion:

$$Q = \theta\frac{\partial}{\partial y} + \tilde{R}\left(x, \eta, \theta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right) + S(x_\bullet, \eta)\frac{\partial}{\partial \theta}.$$

Since $Q^2(y) = Q(\theta) = 0$, we have $S(x_\bullet, \eta_\bullet) = 0$ and therefore:

$$Q = \theta\frac{\partial}{\partial y} + \tilde{R}\left(x, \eta, \theta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right)$$

Since $\theta^2 = 0$, we have:

$$\tilde{R}\left(x, \eta, \theta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right) = A\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right) + \theta B\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right)$$

so that

$$Q = \theta\left(\frac{\partial}{\partial y} + B\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right)\right) + A\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right)$$

There exists local coordinates $(y', x'_\bullet, \eta')$ leaving θ untouched, where

$$\frac{\partial}{\partial y} + B\left(x, \eta, \frac{\partial}{\partial x_\bullet}, \frac{\partial}{\partial \eta_\bullet}\right) = \frac{\partial}{\partial y'}.$$

In these coordinates, Q reads as follows:

$$Q = \theta\frac{\partial}{\partial y'} + A'\left(x', \eta', y', \frac{\partial}{\partial x'_\bullet}, \frac{\partial}{\partial \eta'_\bullet}, \frac{\partial}{\partial y'}\right).$$

Since $Q^2 = 0$, A' does not depend on y', and:

$$Q = \theta\frac{\partial}{\partial y'} + A''\left(x', \eta', \frac{\partial}{\partial x'_\bullet}, \frac{\partial}{\partial \eta'_\bullet},\right) + T(x', \eta')\frac{\partial}{\partial y'}.$$

We now replace θ by $\theta' = \theta + T(x', \eta')$. Since $(\theta + T(x', \eta')) = Q(y') = \theta'$, we have $A''\left(x', \eta', \frac{\partial}{\partial x'_\bullet}, \frac{\partial}{\partial \eta'_\bullet}, \frac{\partial}{\partial \theta'_\bullet}\right)\theta' = 0$, so that A'' has no component in $\frac{\partial}{\partial \theta'_\bullet}$ and the vector field Q has the desired form in these coordinates. This completes the proof. ∎

Proof. [Proof of Theorem 4.2] The theorem is now an immediate consequence of Proposition 4.6, upon making a finite recursion until the corresponding anchor map vanishes. ■

4.2. **Examples and non-examples.**

Example 4.7. Theorem 4.2, when applied to a Lie algebroids, gives back a classical result [9], which itself is similar to Weinstein splitting theorem for Poisson manifolds [39]. For Lie ∞-algebroids, Theorem 4.2 gives back a similar statement in [6].

Example 4.8. For a Koszul-Tate resolution, Theorem 4.2 does not give any interesting result, since the anchor is zero at every point of the zero locus.

Example 4.9. For a positively graded Q-manifold over a manifold M_0, the image of the anchor map

$$\rho\colon \Gamma(E_{-1}) \longrightarrow \mathfrak{X}(M_0)$$

is a singular foliation in the sense of [2], i.e. a locally finitely generated $\mathcal{C}^\infty(M_0)$-sub-module of $\mathfrak{X}(M_0)$ closed under Lie bracket. For $\mathbb{Z}^*$-graded Q-manifold with splitting, whose dual Lie ∞-algebroid with anchor maps $(\rho_n)_{n\geq 1}$, it is natural to ask if

$$\bigoplus_{n\geq 1} \rho_n(\Gamma(S^n \oplus_{i\in\mathbb{Z}} E_i)_{-1})$$

is still a singular foliation. The answer is *no*: it is certainly a $\mathcal{C}^\infty(M_0)$-sub-module of $\mathfrak{X}(M_0)$, but, even when it is locally finitely generated, it may not be stable under Lie bracket. Here is a class of counter-examples: Let M_0 be a manifold, X_1, X_2 vector fields such that $[X_1, X_2]$ is not in the $\mathcal{C}^\infty$-module generated by X_1, X_2, let θ_1, θ_2, η be additional variables of respective degrees 2, 2 and -1, and consider

$$Q = \eta\theta_1 X_1 + \eta\theta_2 X_2.$$

It is straightforward to check that Q is a degree $+1$ vector field squaring to zero. The 2-ary anchor map is not zero and its image is the $\mathcal{C}^\infty(M_0)$ module generated by X_1, X_2, which generated a $\mathcal{C}^\infty(M_0)$-module; by assumption it is not stable under Lie bracket.

Example 4.10. Here is an example of a Q-manifold with a splitting, whose 2-ary anchor is not valued in vector fields tangent to the zero locus:

$$Q = (x - \epsilon\zeta)\frac{\partial}{\partial\eta} + \zeta\xi\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial\epsilon},$$

where x is a degree 0 variables and $\eta, \zeta, \xi, \epsilon$ are variables of respective degrees $-1, 3, -2, -3$.

Conclusion / perspectives

As mentioned in the introduction, the results of [20] on the $\mathbb{Z}$-graded manifolds and the technique of filtrations of functional spaces open a way to understanding the form of various geometric and algebraic structures on them. This permitted for example, to extend the results of [16] to the honest $\mathbb{Z}$-graded case and develop them in [21]. In the current paper we have added an important ingredient to the picture – a Q-structure – describing thus the normal form of differential $\mathbb{Z}^*$-graded manifolds. Our common thread is that "for a $\mathbb{Z}^*$-graded Q-manifold, only the zero locus of the curvature matters": Proposition

2.3 should be understood as meaning that outside the zero locus of their curvatures, $\mathbb{Z}^*$-graded Q-manifolds have a very trivial structure; then Theorem 3.21 makes more precise this general idea, by stating that positive part of a Q-manifold over its zero locus is the only piece that matters when its negative part is a Koszul-Tate resolution; and last, Theorem 4.2 adds another layer to the same general idea, by stating that, at a point in the zero locus, the anchor map and its transverse Q-manifold are the only two non-trivial pieces of information.

Notice that we have not dealt with the problem of linearization of the vector field Q, which is addressed e.g. in [32, 1] and linked in this context to deformation problems: this is a related but different subject.

The current work has initiated several follow-up discussions on other closely related subjects. In particular, the technique was relevant to address the Strobl conjecture on the normal form of singular foliations over singular spaces, which continued in [26] It also has some non-trivial links to the constructions from the "derived world": [4, 13].

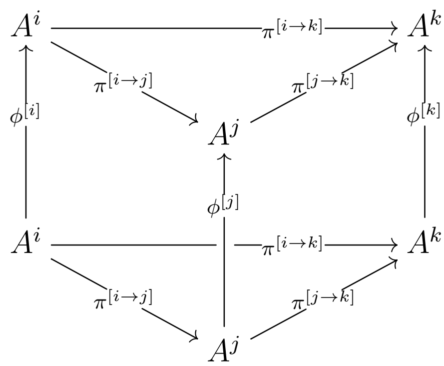
On top of the pure mathematical significance of the above results we expect them to have straightforward consequences for gauge theories. According to [11], under rather natural assumptions one can read off a Q-structure from the equations governing the theory. This language is also widely used for various quantization problems. Then, as explained in [22], a lot of information can be encoded in the language of mappings between Q-manifolds: the equations of motion (i.e. extrema of the functional describing the model) correspond to Q-morphisms, and gauge transformations (symmetries) to Q-homotopies. In this setting reducing a Q-structure to a (simple) canonical form by a homotopy would mean gauge fixing in an intelligent way.

Appendix A. Projective systems of graded algebras

We call *projective system of graded algebras* a pair made of a sequence $(A^i)_{i\in\mathbb{N}}$ of graded algebras, and a family of degree 0 graded algebra morphisms $\pi^{[i\to j]}\colon A^i \to A^j$, defined for all integers $i \geq j$, subject to the two following conditions: $\pi^{[i\to i]} = \mathrm{id}_{A^i}$ and

$$\pi^{[j\to k]} \circ \pi^{[i\to j]} = \pi^{[i\to k]}, \ \ \forall i \geq j \geq k.$$

An *endomorphism of projective graded algebras* is a family $(\phi^{[i]})_{i\in\mathbb{N}}$ of degree 0 algebra endomorphisms $\phi^{[i]} : A^i \to A^i$, defined for all $i \in \mathbb{N}$ such that $\phi^{[j]} \circ \pi^{[i\to j]} = \pi^{[i\to j]} \circ \phi^{[i]}$ for all $i \geq j$. The following diagram recapitulates the above commutativity properties for all $i \geq j \geq k$:



We define the *graded projective limit* A^∞ of a projective system of algebras[9] to be the graded algebra whose component of degree $c \in \mathbb{Z}$ is made of all collections $i \mapsto a^i \in A^i_c$ such that $\pi^{[i\to j]}(a^i) = a^j$ for all $i \geq j$. By assigning to such a collection its i-th component, one defines, for all $i \in \mathbb{N}$, graded algebra morphisms $\pi^{[\infty\to i]} \colon A^\infty \to A^i$ that satisfy:

$$\pi^{[j\to k]} \circ \pi^{[\infty\to j]} = \pi^{[\infty\to k]}, \ \ \forall j \geq k.$$

For any morphism of projective graded algebras $(\phi^{[i]})_{i\in\mathbb{N}}$, there exists a unique graded algebra endomorphism $\phi^{[\infty]} \colon A^\infty \to A^\infty$ such that $\phi^{[i]} \circ \pi^{[\infty\to i]} = \pi^{[\infty\to i]} \circ \phi^{[\infty]}$.

$$\begin{array}{ccc} A^\infty & \xrightarrow{\pi^{[\infty\to i]}} & A^i \\ {\scriptstyle \phi^{[\infty]}}\Big\uparrow & & \Big\uparrow{\scriptstyle \phi^{[i]}} \\ A^\infty & \xrightarrow{\pi^{[\infty\to i]}} & A^i \end{array}$$

We call $\phi^{[\infty]} \colon A^\infty \to A^\infty$ the *graded projective limit of* $(\phi^{[i]})_{i\in\mathbb{N}}$.

Proposition A.1. *Let $\left(A^i, \pi^{[i\to j]}\right)$ be a projective system of graded algebras. For any family $(\phi_N)_{N\in\mathbb{N}}$ of degree 0 endomorphisms of the latter such that $\phi_N^{[i]} = \mathrm{id}_{A^i}$ for all $N \geq i$, the sequence of degree 0 algebra endomorphisms defined for all $i \in \mathbb{N}$ by*

$$\begin{array}{rccl} \psi^{[i]} : & A^i & \to & A^i \\ & a & \mapsto & \cdots \circ \phi_3^{[i]} \circ \phi_2^{[i]} \circ \phi_1^{[i]}(a) = \\ & & & \phi_i^{[i]} \circ \cdots \circ \phi_1^{[i]}(a) \ \ \textit{(by assumption)} \end{array}$$

is an endomorphism of projective systems of graded algebras.

The projective limit $\psi^{[\infty]} \colon A^\infty \to A^\infty$ must be understood as the infinite composition of all the $(\phi_i)_{i\in\mathbb{N}}$, it will therefore be denoted by $\bigcirc_{i\uparrow\in\mathbb{N}} \phi_i$ or $\prod_{i\uparrow\in\mathbb{N}} \phi_i$, where by "$i \uparrow\in \mathbb{N}$" we mean that the composition is computed over the increasing index i.

[9] In the presented construction, the system of algebras can be labeled by any partially ordered set I. For the particular case of $I = \mathbb{N}$, it simplifies, producing a tower of algebra morphisms $A^1 \leftarrow A^2 \leftarrow \cdots$. Then the compatible collection $i \mapsto a^i$ will simply be a sequence $\{a_i\}_{i\in\mathbb{N}}$, where a_i is an element of A^i, such that a_{i+1} maps to a_i for all $i \in \mathbb{N}$. Also, we think that introducing $\pi^{[\infty\to i]}$ is important for a good conceptual understanding, and therefore prefer to present the construction with this generality.

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Alexei Kotov: Faculty of Science, University of Hradec Kralove, Rokitanskeho 62, Hradec Kralove 50003, Czech Republic
Email address: `oleksii.kotov@uhk.cz`

Camille Laurent-Gengoux: Université de Lorraine, CNRS, IECL, F-57000 Metz, France
Email address: `camille.laurent-gengoux@univ-lorraine.fr`

Vladimir Salnikov: LaSIE – CNRS & La Rochelle University, Av. Michel Crépeau, 17042 La Rochelle Cedex 1, France
Email address: `vladimir.salnikov@univ-lr.fr`