A few remarks on penalty and penalty-duality methods in fluid-structure interactions

Philippe Destuynder, Erwan Liberge

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Few remarks on penalty and penalty-duality methods in fluid-structure interactions

Philippe Destuynder* and Erwan Liberge*

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* Laboratoire LaSIE
Université de La Rochelle
Avenue Michel Crépeau, 17042 La Rochelle France
philippe.destuynder@univ-lr.fr,   erwan.liberge@univ-lr.fr

Abstract

In fluid-structure interaction problems, many people use a penalty method for positioning the structure inside the fluid. This is usually performed by considering that the fluid is very stiff or/and very heavy at the place occupied by the structure. These methods are very convenient for the programming point of view but can lead to ill conditioned operators. This is a drawback in the numerical solution methods. In particular the forces applied to the structure by the surrounding flow are not always accurately estimated because the penalty parameter -which is a very large number- appears in their expressions. We suggest in this paper a mathematical analysis of the difficulties encountered and we discuss how the penalty-duality method of D. Bertsekas can be an interesting alternative to overcome them.

1 Introduction

Let us consider -for sake of brevity- a two dimensional bounded and connected open set denoted by Ω with boundary Γ. The unit normal to Γ outward Ω is denoted by ν. Inside Ω there is an open subset S with boundary ∂S. The complementary open set of $\overline{S}$ in Ω is $\Omega_S = \Omega \setminus \overline{S}$. The set $\overline{S}$ is occupied by a structure that we assume to be rigid or/and heavy for sake of simplicity. But there is no difficulty for extending our discussion to a flexible structure. The open set Ω_S is occupied by a viscous fluid that we assume to be incompressible in order to close the set of equations modelling the flow. The open set S is moving with the displacement of the structure. Hence a new discretization of Ω_S would be necessary in case where one chooses to model separately the two media. Furthermore the transport of the physical field requires some special transformation as a local (or global) Euler-Lagrange parametrization. In addition, the approximation and the transport of the forces (or stresses) interacting between the two media are a tough problem in terms of precision.

An alternative which has been introduced by several authors [1].[12], consists in considering that the two media obey to the same equilibrium laws but differ by different constitutive relationships. More precisely, the coefficients of the constitutive equation are very large in order to prescribe a rigid body motion in the open set occupied by the structure. Another possibility consists in forcing the velocity field of the fluid to be equal to a rigid body
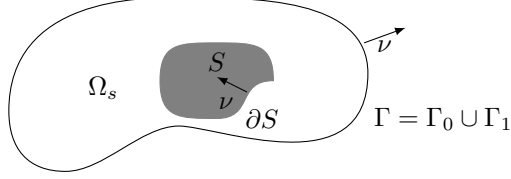
Figure 1: Schematic description of fluid structure interaction problem

one (the one of the structure) by adding a penalty term using the $L^2(S)$-norm [4].[29].[5]. The advantage of these approaches is to work on a single mesh (but requires some interpolations as far as the structure is moving on this mesh and its boundary has no reason to respect the vertices of the mesh. The second possibility is a very promising method for particles methods and lattice Boltzmann modeling [4].[29].[5]. Vortex methods are also well adapted to this strategy [28].[13].[14].

Several authors [33].[35].[26] have suggested to take into account the penalty term only on the boundary of the structure. But the question of the ill conditioning and the one of the mapping between the global mesh and this boundary should be handled in order to improve the strategy, mainly in a multiprocessor programming. Our purpose in this paper is to discuss mathematically the penalty methods and the penalty-duality method introduced by D. Bertsekas [9] with these different possibilities concerning the penalty term. This method has also been developed by M. Fortin and R. Glowinski [22] for several engineering problems. More recently a duality method has also been used for fluid-structure problems by L. Formaggia- A. Quarteroni - A. Veneziani [21] and N.D. dos Santos [34]. But the method is from the one used in [9] and [22] and the present paper. We chose to remain at the theoretical level in order to point out the mathematical aspects of the problems. Furthermore, our analysis is performed for a given position of the structure and with a given rigid body velocity of it. In a dynamical model one could consider that it is the case when the classical algorithm (transport prediction and diffusion correction) is used (see P.A. Raviart and V. Girault [25]. Concerning the movement of the structure it is assumed to be sufficiently slow in order to adopt an explicit scheme for its displacements (small reduced frequencies approximation [23].[16]).

Let us first state few notations used in the following. The velocity of the fluid particles is denoted by $\mathbf{u} = \{u_i\}$, $i = 1, 2$ (the extension to 3D is not a difficulty for the theoretical approach as far as one restricts the analysis to the Stokes model, but it is a real one concerning the implementation). The strain ratio is $\gamma(\mathbf{u}) = \{\gamma_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)\}$ and the stress ratio is (div($\mathbf{u}$) = 0):

$$\sigma_{ij} = 2\mu\gamma_{ij}(\mathbf{u}),$$

μ being the viscosity of the fluid. The unit normal to ∂S inward S is also denoted by ν and the movement of the solid is represented by its rigid body velocity ($\mathbf{k}$ is the unit vector normal to the plan containing Ω; r is the velocity rotation around $\mathbf{k}$ and $\mathbf{a}$ the velocity of point O). Hence the the velocity of the structure at point $\mathbf{x}$ is:

$$\mathbf{d} = \{d_i\} = \mathbf{a} + r\mathbf{k} \wedge \mathbf{Ox} = \left\{ \begin{array}{c} a_1 - rx_2 \\ \\ a_2 + rx_1 \end{array} \right\}, \quad \mathbf{x} = (x_1, x_2) \text{ and } \mathbf{a} = (a_1, a_2). \quad (1)$$

The kinematical continuity between the fluid and the structure is traduced by the fol-

lowing relation:
$$\mathbf{u} = \mathbf{a} + r\mathbf{k} \wedge \mathbf{Ox} \text{ on } \partial S. \qquad (2)$$

The flow model inside Ω_S consists in finding $\mathbf{u}$ such that (Γ_0 and Γ_1 being the two distinct components of Γ where respectively the flow velocity -respectively the normal stress- is given and p is the static pressure):

$$\begin{cases} \sigma(\mathbf{u}) = \mu\gamma(\mathbf{u}), \ \mathbf{div}(\sigma) - \mathbf{grad}(p) = 0 \text{ in } \Omega_S, \\ \\ \mathbf{div}(\mathbf{u}) = 0 \text{ in } \Omega_S, \\ \\ \mathbf{u} = \mathbf{u}_0 \text{ on } \Gamma_0 \subset \Gamma, \ \sigma.\nu - p\nu = \mathbf{g} \text{ on } \Gamma_1 \subset \Gamma, \\ \\ \text{and finally the condition stated in equation (2) on } \partial S. \end{cases} \qquad (3)$$

This a quasi-static flow model (following the terminology of aeroelasticity and hydroelasticity [23].[16]) because the inertia forces aren't taken into account in the flow.

The equations of the model for the structure are the following ones (let us notice that: $J_O = J_G + M||\mathbf{OG}||^2$ where J_G is the inertia around the center of mass G and J_O the one around the point O and finally the dot stands for the scalar product or more generally for the matrix product):

$$\begin{cases} M\dot{\mathbf{a}} + M\dot{r}\mathbf{k} \wedge \mathbf{OG} + Mr\mathbf{k} \wedge \mathbf{a} - Mr^2\mathbf{OG} = \displaystyle\int_{\partial S} p\nu - \sigma.\nu, \\ \\ J_O\dot{r} + M(\dot{\mathbf{a}}, \mathbf{k}, \mathbf{OG}) + Mr(\mathbf{a}.\mathbf{OG}) + 2Mr(\mathbf{k} \wedge \mathbf{OG})(\mathbf{k} \wedge \mathbf{a}) \\ \\ \qquad\qquad\qquad = \displaystyle\int_{\partial S} (p\nu - \sigma.\nu) \wedge \mathbf{Ox}, \\ \\ +\text{initial conditions on } \mathbf{a} \text{ and } r. \end{cases} \qquad (4)$$

Remark 1. In fact, we have considered in the previous modeling, that the movement of the structure is quite slow compared to the velocity necessary for establishing the permanent flow. In other words, it is the quasic-static approximation corresponding to low reduce frequencies [23].[16]. Therefore the problem to be solved here is a simple first order differential equation in $\mathbf{a}$ and r coupled with a static linear partial differential equation modeling the flow around the moving structure. Therefore we focus on the solution method for this last problem explained in equation (3) assuming that ∂S, $\mathbf{u}_0$ and $\mathbf{g}$ are given. □

2 The initial formulation for the flow

First of all let us introduce the problem (3) as an optimization one. We introduce the functional space for the velocity field of the fluid:

$$V_0 = \{\mathbf{v} = \{v_i\}, \ i = 1, 2, \ v_i \in H^1(\Omega_S), \ \mathbf{div}(\mathbf{v}) = 0 \text{ in } \Omega_S, \ \mathbf{v} = 0 \text{ on } \Gamma_0\}. \qquad (5)$$

Then we set ($\sigma.\gamma = \sum_{i,j \in \{1,2\}} \sigma_{ij}\gamma_{ij}$):

$$\forall \mathbf{v} \in V_0, \ J(\mathbf{v}) = \frac{1}{2}\int_{\Omega_S} \sigma(\mathbf{v}).\gamma(\mathbf{v}) - \int_{\Gamma_1} \mathbf{g}.\mathbf{v} \qquad (6)$$

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It is classical that equations (3) are equivalent to:

$$\min_{\substack{\mathbf{v} \in V_0 \ \mathbf{v} = \mathbf{a} + r\mathbf{k} \wedge \mathbf{O}x \ \text{on} \ \partial S}} J(\mathbf{v}). \tag{7}$$

Existence and uniqueness of the solution -say $\mathbf{u}$- are also well known for Stokes model(see for instance [25]).

Remark 2. The most popular method for solving (3) is based on the mixed formulation where both the velocity and the pressure are unknowns. The results are very good and an extension with the acceleration term:

$$\varrho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \nabla \mathbf{u} \right],$$

works perfectly. But, this formulation is restricted to incompressible (or slightly compressible) flows. The details are given for instance in [25]. □

3 The extension to the full domain using penalty methods

3.1 Definitions of several penalty models

The velocity $\mathbf{u}_0$ on the structure is defined by a rigid body description. Such velocity fields satisfy the relation:

$$\gamma_{ij}(\mathbf{u}_0) = 0 \quad \forall i, j \in \{1, 2\}. \tag{8}$$

The idea of the penalty method in this case, consists in replacing problem (7) by the next one ($\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ $\varepsilon > 0$):

$$\begin{cases} \displaystyle \min_{\mathbf{v} \in W_0} J^{\varepsilon}(\mathbf{v}) \\[2mm] \text{where:} \\[2mm] J^{\varepsilon}(\mathbf{v}) = \frac{1}{2} \left[\int_{\Omega} \sigma(\mathbf{v}).\gamma(\mathbf{v}) + \frac{\alpha}{\varepsilon} \int_{\partial S} ||\mathbf{v} - \mathbf{u}_0||^2 + \frac{\beta}{\varepsilon} \int_{S} \sigma(\mathbf{v}).\gamma(\mathbf{v}) + \frac{\gamma}{\varepsilon} \int_{S} ||\mathbf{v} - \mathbf{u}_0||^2 \right] \quad (9) \\[2mm] \text{and:} \\[2mm] W_0 = \left\{ \mathbf{v} = \{v_i\}, \ i = 1, 2, \ v_i \in H^1(\Omega), \ \mathbf{div}(\mathbf{v}) = 0 \ \text{on} \ \Omega, \ \mathbf{v} = 0 \ \text{on} \ \Gamma_0 \right\}. \end{cases}$$

Here again the existence and uniqueness of a solution are standard for a given set of the parameters $\alpha, \beta, \gamma, \varepsilon$. The solution of (9) is a function of α, β, γ and mainly of ε. Our purpose in this section, is to analyse the behaviour of this solution denoted by $\mathbf{u}^{\varepsilon}$ for several choices of α, β and γ, when ε tends to zero. Let us make explicit the variational formulation of (9). The solution $\mathbf{u}^{\varepsilon}$ is characterized by:

$$\begin{cases} \mathbf{u}^{\varepsilon} \in W_0 \ \text{such that} \ \forall \mathbf{v} \in W_0 : \\[2mm] \displaystyle \int_{\Omega} \sigma(\mathbf{u}^{\varepsilon}).\gamma(\mathbf{v}) + \frac{\alpha}{\varepsilon} \int_{\partial S} (\mathbf{u}^{\varepsilon} - \mathbf{u}_0).\mathbf{v} + \frac{\beta}{\varepsilon} \int_{S} \sigma(\mathbf{u}^{\varepsilon}).\gamma(\mathbf{v}) \\[4mm] \displaystyle \hspace{4cm} + \frac{\gamma}{\varepsilon} \int_{S} (\mathbf{u}^{\varepsilon} - \mathbf{u}_0).\mathbf{v} = \int_{\Gamma_1} \mathbf{g}.\mathbf{v} \end{cases} \quad (10)$$

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Remark 3. In the papers by Benamour-Liberge-Béghein [4].[29].[5], the authors set $\alpha = 0$ and also $\beta = 0$ but $\gamma \neq 0$. This is a very convenient strategy because one can use the mass matrix which is required for the dynamic analysis (the mass density for this new term is artificially equal to γ/ε in S). The solution $\mathbf{u}^\varepsilon$ and also the normal stress associated should be continuous across the boundary ∂S and this last condition (on the normal stresses) is not well satisfied localy for $\varepsilon \simeq 0$. The reason is that the bilinear form involved in the limit model for $\alpha = \beta = 0$, which couples $\mathbf{u}^0$ on Ω_S and $\mathbf{u}^1$ on S, is only coercive on the space:

$$W_\gamma = \{\mathbf{v} \text{ s.t. } \operatorname{div}(\mathbf{v}) = 0 \text{ in } \Omega \text{ and } \mathbf{v}_{|\Omega} \in [H^1(\Omega_S)]^2, \ \mathbf{v}_{|S} \in [L^2(S)]^2\}. \tag{11}$$

It is a larger space than W_0 and such that only the continuity of the normal velocity $\mathbf{v}.\nu$ can be verified in the space $H^{-1/2}(\partial S)$ using the trace Theorem (see for instance [31]) which is a weak result, mainly concerning the evaluation of the stress due to the fluid and applied to the structure. Furthermore, the tangential component of $\mathbf{v}_{|S}$ on ∂S doesn't exist in such a space. Nevertheless this strategy is very tricky for the numerical implementation and can lead to nice and cheap evaluation of the velocity field. Therefore we discuss it in the following in order to check which kind of information can be lost concerning the tangential component of the normal stress along ∂S.

If $\beta \neq 0$ whatever would be α and γ, one can can ensure the continuity of the two components of the velocity field $\mathbf{u}^\varepsilon$ solution of (9) when $\varepsilon \to 0$. Hence only the rigid body motion of the open set S is taken into account. Let us point out that if $\alpha = \gamma = 0$ nothing guarantees that it will be the one prescribed by the equations of the mechanics. In fact $\mathbf{u}_0$ has disappeared from the penalty model and this statement is propped by the numerical tests of section 4.

A discussion is carried out at subsection 3.2 in order to explain the drawbacks of this nice strategy which is nevertheless used for convenience in the implementation in existing software by many of authors. □

Remark 4. The implementation of the penalty term on S for $\beta > 0$ is easy to do as far it only consists in choosing a new expression for the viscosity coefficient of the fluid. We set μ in Ω_S and $\mu(1 + \frac{\beta}{\varepsilon})$ in S. □

Remark 5. In a numerical implementations, one needs to use two mappings because the movement of the structure does respect the sides of the mesh used in the approximation of the fluid movement. One -say R^h- maps the position of ∂S on the mesh and the other -say P^h- maps the mesh of Ω on ∂S. But this embedding problem is fully and accurately solved in a practical formulation by C. Farhat and his coworkers [10]-[27]. A mathematical formulation of this arbitrary Euler-Lagrange parametrization is briefly discussed in [15] using domain derivative tools for improving the normal stresses approximation between the fluid and the structure. □

Remark 6. Let us justify on a very simple example why we claim that penalty methods can lead to ill conditioning. let consider the optimization problem:

$$\begin{cases} \min_{y \in \mathbb{R}^2, \, c.y=0} \dfrac{1}{2} Ay.y - b.y, \\[2mm] \text{with:} \\[2mm] A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{cases} \tag{12}$$

The solution is easily computed and is: $\tilde{y}_1 = .5$ and $\tilde{y}_2 = 0$. A basic penalty method can be defined as follows for any $\varepsilon > 0$:

$$\min_{y \in \mathbb{R}^2, \, c.y=0} \frac{1}{2} A^\varepsilon . y - b.y \quad \text{where: } A^\varepsilon = \begin{pmatrix} 2 & -1 \\ \\ -1 & 1 + \frac{1}{\varepsilon} \end{pmatrix}, \tag{13}$$

with the solution $\tilde{y}_1^\varepsilon = 1, \tilde{y}_2^\varepsilon = \frac{\varepsilon}{1+\varepsilon}$. But the condition number of the matrix A^ε is: $C(A^\varepsilon) = \frac{1+\varepsilon}{\varepsilon} \rightarrow_{\varepsilon \to 0} \infty$ which justify the ill conditioning of the matrix associated to the penalty method and there fore the high sensitivity of the penalized solution, to truncation of digits or approximations which can mainly appears in structural boundary approximation for 2D or 3D models.

3.2 The asymptotic analysis

Let set *a priori*:
$$\mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \dots \tag{14}$$

By introducing this expression into the variational formulation (10) and by equating the terms of same power in ε one obtains the following necessary conditions:

$$\begin{cases} \text{i) terms of order } \varepsilon^{-1} : \; \forall \mathbf{v} \in W_0, \; (\text{ let us notice that } \sigma(\mathbf{u}_0) = 0) \\[2mm] \alpha \int_{\partial S} (\mathbf{u}^0 - \mathbf{u}_0).\mathbf{v} + \beta \int_S \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) + \gamma \int_S (\mathbf{u}^0 - \mathbf{u}_0).\mathbf{v} = 0, \\[4mm] \text{ii) terms of order zero: } \forall \mathbf{v} \in W_0, \\[2mm] \int_\Omega \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) + \alpha \int_{\partial S} \mathbf{u}^1.\mathbf{v} + \beta \int_S \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) + \gamma \int_S \mathbf{u}^1.\mathbf{v} = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}, \\[6mm] \text{iii) terms of order } \varepsilon : \; \forall \mathbf{v} \in W_0, \\[2mm] \int_\Omega \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) + \alpha \int_{\partial S} \mathbf{u}^2.\mathbf{v} + \beta \int_S \sigma(\mathbf{u}^2).\gamma(\mathbf{v}) + \gamma \int_S \mathbf{u}^2.\mathbf{v} = 0, \\[4mm] \dots \end{cases} \tag{15}$$

Setting on S:
$$\mathbf{u}^0 = \mathbf{u}_0 + \mathbf{u}^{00}, \; \mathbf{u}^{00} \in W_0,$$

with the condition $\mathbf{u}^{00} = 0$ on ∂S, equation i) of (15) implies that:

$$\begin{cases} \text{if } \alpha, \beta > 0 \text{ but } \gamma \geq 0 : \\[2mm] \qquad => \mathbf{u}^0 = \mathbf{u}_0 \text{ on } \partial S \text{ and } \gamma_{ij}(\mathbf{u}^{00}) = 0 \text{ in } S => \mathbf{u}^{00} = 0 \text{ on } S, \\[2mm] \text{if } \beta = 0 \text{ but } \gamma > 0 \text{ one has for any } \alpha \geq 0 : \\[2mm] \qquad => \mathbf{u}^{00} = 0. \end{cases} \tag{16}$$

Therefore $\mathbf{u}^0$ is a rigid body velocity on S and equal to $\mathbf{u}_0$ as far as α or γ is (are) strictly positive. But it should be noticed that this last statement is valid only because $\alpha > 0$ or $\gamma > 0$. For $\alpha = \gamma = 0$ one could only claim that $\mathbf{u}^0$ is a rigid body velocity on S but not necessarily equal to $\mathbf{u}_0$. At this step, the case $\alpha = \beta = 0$ seems to be the most convenient, but the following contradicts this first conclusion.

We now introduce a new functional space by:

$$K_0 = \{\mathbf{v} \in W_0,\ \mathbf{v} = 0 \text{ in } S\}. \tag{17}$$

From equation ii) of (15), one deduces that:

$$\forall \mathbf{v} \in K_0,\ \int_{\Omega_S} \sigma.(\mathbf{u}^0).\gamma(\mathbf{v}) = \int_{\Gamma_1} \mathbf{g}.\mathbf{v},\ \text{ and if } \alpha^2 + \gamma^2 > 0\ => \mathbf{u}^0 = \mathbf{u}_0 \text{ on } \partial S. \tag{18}$$

The element $\mathbf{u}^0 \in K_0$ is now perfectly defined and is identical on Ω_S to the initial solution $\mathbf{u}$ characterized at (3).

Remark 7. If $\alpha = \gamma = 0$ and $\beta > 0$ one has only:

$$\gamma_{ij}(\mathbf{u}^0) = 0 \text{ in } S => \mathbf{u}^0 = a + r\mathbf{k} \wedge \mathbf{Ox} \text{ in } S. \tag{19}$$

The term $\mathbf{u}^0$ belongs to the following closed subspace of W_0 defined by:

$$RBM = \{\mathbf{v} \in W_0,\ \gamma(\mathbf{v}) = 0 \text{ in } S\}. \tag{20}$$

Hence in this case ($\alpha = \gamma = 0$) the term $\mathbf{u}^0 \in RBM$ is the unique solution of:

$$\forall \mathbf{v} \in RBM,\ \int_{\Omega} \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}\quad . \tag{21}$$

But the rigid body motion on S is derived from the solution $\mathbf{u}^0$ and is no more the one that has been prescribed from the movement of the structure. This is the fundamental difference that suggests to avoid the choice $\alpha = 0$ or $\gamma = 0$. In fact the choice $\beta = 0$ is another possibility which will appear as a smart possibility in the discussion contained in the next remark 9, even if it has its own drawbacks concerning the regularity if the boundary ∂S is not smooth enough as we mention in the following. □

Remark 8. Let us now consider the case where $\alpha > 0$ and $\beta = \gamma = 0$. This is very interesting situation from the practical point of view, because we only have to manage the position of the boundary ∂S inside the mesh which can be unchanged during the displacement of the structure (full eulerian representation of the fluid). Let us also underline that this method is just for the quasi-static case. If inertia terms are introduced in the fluid then it doesn't work in this case. But even for quasi-static cases (no inertia terms), this assertion could be false for a multiprocessor programming where each processor could be idling when waiting for the informations concerning this boundary ∂S. At the order minus one in ε in the asymptotic expansion, one only gets the condition $\mathbf{u}^0 = \mathbf{u}_0$ on ∂S. But at the order zero in ε one has:

$$\forall \mathbf{v} \in W_0,\ \int_{\Omega} \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) + \alpha \int_{\partial S} \mathbf{u}^1.\mathbf{v} = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}. \tag{22}$$

Let us define three closed subspaces of W_0 by:

$$\begin{cases} W_{00} = \{v \in W_0, \ \mathbf{v} = 0 \text{ on } \partial S\}, \\[2mm] V_{00\Omega_S} = \{\mathbf{v} \in V_0 \text{ and } \mathbf{v} = 0 \text{ on } \partial S, \ (\mathbf{v} = 0 \text{ on } S)\}; \\[2mm] V_{00S} = [H_0^1(S)]^2. \end{cases} \tag{23}$$

It is worth to notice that one can identify W_{00} with the space $V_{00\Omega} \times V_{00S}$.

By restricting the virtual velocity $\mathbf{v}$ to the space W_{00} ($\mathbf{u}^0 = \mathbf{u}_0$ (the rigid body velocity) on ∂S) one obtains two distinct sub-problems; one is set on Ω_S and the other on S. They are given hereafter:

$$\begin{cases} \mathbf{u}^0_{|\Omega_S} \in V_0, \forall \mathbf{v} \in V_{00\Omega_S} : \ \int_{\Omega_S} \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}, \ \mathbf{u}^0 = \mathbf{u}_0 \text{ on } \partial S, \\[4mm] \mathbf{u}^0_{|S} \in V_{00S}, \ \forall \mathbf{v} \in V_{00S}, \ \int_S \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) = 0, \ \mathbf{u}^0 = \mathbf{u}_0 \text{ on } \partial S. \end{cases} \tag{24}$$

The system of equations (24) leads to two independent problems for which the solutions are uniquely defined respectively on Ω_S and on S. The solution $\mathbf{u}^0$ on Ω_S is precisely the one of the initial model we started from (3). The solution $\mathbf{u}^0$ on S can easily be characterized. Because the solution is unique and because the rigid body $\mathbf{u}_0$ is a solution, it is the right one. Hence $\mathbf{u}^0$ is exactly the solution in the full domain Ω of the initial problem.

Nevertheless, it should be underlined that in this case ($\beta = \gamma = 0$ and $\alpha > 0$) the penalty term is reduced to a boundary term implying the $L^2(\partial S)$ norm and therefore the continuous model is perfectly defined. The numerical aspects are easy to handle if we use a separate modeling of ∂S (we do not need to divide the elements at the boundary between S and Ω_S). But the condition number of the discretized model remains an important question in the solution method compared to the case where $\alpha > 0$ and $\beta > 0$. □

Remark 9. In this remark we discuss the case where $\alpha = \beta = 0$ and $\gamma > 0$. The first equation (15) leads directly to $\mathbf{u}^0 = \mathbf{u}_0$ in S. Therefore, from the second equation (15) $\mathbf{u}^0$ should connected to an element $\mathbf{u}^1$ such that:

$$\forall \mathbf{v} \in W_0, \ \int_\Omega \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) + \gamma \int_S \mathbf{u}^1.\mathbf{v} = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}. \tag{25}$$

Because we proved that $\mathbf{u}^0 = \mathbf{u}_0$ in S, the characterization of $\mathbf{u}^0$ on Ω_S is obtained by solving the equation (which has a unique solution):

$$\mathbf{u}^0_{|\Omega_S} \in V_0, \forall \mathbf{v} \in V_{00\Omega_S}, \ \int_{\Omega_S} \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}, \ \mathbf{u}^0 = \mathbf{u}_0 \text{ on } \partial S. \tag{26}$$

Consequently, the term $\mathbf{u}^1$ should satisfy in S, the relation:

$$\mathbf{u}^1 = 0, \tag{27}$$

which is not compatible with equation (25) which leads to (ν is here the unit normal along ∂S inward S):

$$\forall \mathbf{v} \in W_0, \ \int_{\partial S} (\sigma(\mathbf{u}^0).\nu).\mathbf{v} + \gamma \int_S \mathbf{u}^1.\mathbf{v} = 0. \tag{28}$$

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In fact it is impossible to compute $\mathbf{u}^1$ which would appear as a first order derivative at the origin of $\mathbf{u}^\varepsilon$ with respect to ε. This phenomenon is due to a stiff boundary layer near ∂S which is fully analyzed in the linear case in the book of J.L. Lions on singular perturbations [32]. Using interpolation technics between Hilbert spaces, he proved that one could only hope a fractional derivative order and furthermore in a larger space than the energy one. This result is extended to Navier-Stokes equations in the paper by P. Angot & all [1]. Nevertheless the convergence of $\mathbf{u}^\varepsilon$ to $\mathbf{u}^0$ is true in energy but for the weak topology for $\alpha = \beta = 0$ and $\gamma > 0$ as we proved at Theorem 3.1 at the end of this remark. But the speed of convergence in ε is not obvious and furthermore in a larger space than the energy one. This justifies to use this penalty method in this framework when the programming advantages are clear (multiprocessor programming for instance). A numerical discussion concerning these theoretical results on a very simple model is suggested in section 4. $\quad\square$

Theorem 3.1. *Let $\mathbf{u}^\varepsilon \in W_0$ be solution of (10) and $\mathbf{u}^0$ be the solution of (26) in Ω_S such that $\mathbf{u}^0 = \mathbf{u}_0$ in S. One has:*

$$\lim_{\varepsilon \to 0} ||\mathbf{u}^\varepsilon - \mathbf{u}^0||_{0,\Omega} = 0 \text{ and } ||\mathbf{u}^\varepsilon - \mathbf{u}_0||_{0,S} \le c\sqrt{\varepsilon}. \tag{29}$$

$\square$

Proof. Let set $\mathbf{v} = \mathbf{u}^\varepsilon$ in equation (10). Using Korn inequality [18], one can claim that there exists a constant independent on ε -say $c_0 > 0$- such that:

$$c_0||\mathbf{u}^\varepsilon||_{1,\Omega}^2 + \frac{\gamma}{\varepsilon}||\mathbf{u}^\varepsilon - \mathbf{u}_0||_{0,S}^2 \le \frac{\gamma}{\varepsilon} \int_S \mathbf{u}_0 \mathbf{u}^\varepsilon + \int_{\Gamma_1} \mathbf{g}.\mathbf{u}^\varepsilon. \tag{30}$$

Using the triangular Cauchy-Schwarz[1] inequality and the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\Gamma_1)$, on deduces the existence of another constant independent on ε -say c_1- such that:

$$c_0||\mathbf{u}^\varepsilon||_{1,\Omega}^2 + \frac{\gamma}{\varepsilon}||\mathbf{u}^\varepsilon - \mathbf{u}_0||_{0,S}^2 \le c_1. \tag{31}$$

Therefore one can extract from the sequence $\mathbf{u}^\varepsilon$ a subsequence denoted by $\mathbf{u}^{\varepsilon'}$ and such that:

$$\begin{cases} i) \text{ in } S : \lim_{\varepsilon \to 0} ||\mathbf{u}^\varepsilon - \mathbf{u}_0||_{0,S} = 0 \text{ (strong convergence with the rate } \sqrt{\varepsilon}\text{),} \\[2ex] i) \text{ in } \Omega : \lim_{\varepsilon' \to 0} \mathbf{u}^{\varepsilon'} = \mathbf{u}^* \text{ in } [H^1(\Omega)]^2\text{-weakly; hence } \mathbf{u}^* = \mathbf{u}_0 \text{ in } S. \end{cases} \tag{32}$$

Taking the limit in (10), one characterizes $\mathbf{u}^*$ as the unique solution of:

$$\mathbf{u}^* = \mathbf{u}_0 \text{ on } \partial S \text{ and } \forall \mathbf{v} \in K_0, \int_{\Omega_S} \sigma(\mathbf{u}^*).\gamma(\mathbf{v}) = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}. \tag{33}$$

In other words $\mathbf{u}^* = \mathbf{u}^0$ in all Ω. Because of the uniqueness of $\mathbf{u}^0$, all the sequence $\mathbf{u}^\varepsilon$ converges to $\mathbf{u}^0$ in $H^1(\Omega)$-weakly and strongly in $L^2(S)$. But the embedding from $H^1(\Omega)$ into $L^2(\Omega)$ is compact, hence the convergence of the sequence $\mathbf{u}^\varepsilon$ to $\mathbf{u}^0$ is strong in $[L^2(\Omega)]^2$. $\quad\square$

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[1] $\forall a, b, \zeta \ge 0, \ 2ab \le \zeta a^2 + \frac{1}{\zeta}b^2$

In the following of this section we assume that $\alpha > 0$ and we start with $\beta > 0$ in order to reduce singular perturbations problems in the vicinity of ∂S which would damage the computation of the forces applied to the structure by the fluid. Let us go on with the relation ii) of equation (15) but now for arbitrary element $\mathbf{v}$ in W_0. Because $\mathbf{u}^0$ is perfectly determined, it remains to find an element (not necessarily unique) $\mathbf{u}^1$ in W_0 such that (let us emphasize on the fact that the right-hand side is a linear and continuous form on W_0 without any additional regularity required on $\mathbf{u}^0$):

$$\forall \mathbf{v} \in W_0, \ \alpha \int_{\partial S} \mathbf{u}^1 . \mathbf{v} + \beta \int_S \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) = - \int_{\Omega_S} \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) + \int_{\Gamma_1} \mathbf{g}.\mathbf{v} \ . \quad (34)$$

Or else from an integration by parts (the velocity fields are divergence free):

$$\forall \mathbf{v} \in W_0, \ \alpha \int_{\partial S} \mathbf{u}^1 . \mathbf{v} + \beta \int_S \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) = - \int_{\partial S} (\sigma(\mathbf{u}^0).\nu).\mathbf{v} \ . \quad (35)$$

Let us check that $\mathbf{u}^1$ is perfectly determined by the previous relation on S (but not yet on Ω_S). We denote by $\mathbf{u}^{1p}$ this particular term defined on S. The bilinear form a^1 defined by:

$$\mathbf{u}, \mathbf{v} \in [H^1(S)]^2 \to a^1(\mathbf{u}, \mathbf{v}) = \alpha \int_{\partial S} \mathbf{u}.\mathbf{v} + \beta \int_S \sigma(\mathbf{u}).\gamma(\mathbf{v}), \quad (36)$$

is $[H^1(S)]^2$-coercive (and symmetrical) and therefore, from Lax-Milgram Theorem, one can claim that $\mathbf{u}^1$ is perfectly defined on S as a solution of (35) (using the expression given at (34) the right-hand side is clearly a linear and continuous form on the functional space $[H^1(S)]^2$). It is denoted by $\mathbf{u}^{1p}$ and it also satisfies the incompressibility condition on S.

In order to complete the definition of $\mathbf{u}^1$ on the whole open set Ω, we go to the equations obtained at the order one. Choosing an arbitrary element $\mathbf{v} \in K_0$ (see (41)), one obtains (let us point out that the normal stress due to $\mathbf{u}^1$ is not continuous across ∂S):

$$\begin{cases} \displaystyle\int_{\Omega_S} \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) = 0, \\ \\ \mathbf{u}^1 = \mathbf{u}^{1p} \text{ on } \partial S \text{ and } \mathbf{u}^1_{|\Omega_S} \in V_0. \end{cases} \quad (37)$$

Finally, the term $\mathbf{u}^1$ is defined on the whole open set Ω and because of the continuity of the trace on ∂S, it belongs to the space $[H^1(\Omega)]^2$. Its restriction to S is $\mathbf{u}^{1p}$ characterized at (35) and its restriction to Ω_S is solution of (37). Globally $\mathbf{u}^1$ belongs to the space W_0 because it is divergence free on each open subset of Ω. It should be underlined that no additional regularity is required in the computation of $\mathbf{u}^1$. But this will be different when $\beta = 0$.

Let us finish this partial analysis of an assumed asymptotic expansion of $\mathbf{u}^\varepsilon$, by a partial characterization of $\mathbf{u}^2$ on S denoted by $\mathbf{u}^{2p}$. Once $\mathbf{u}^1$ is known one should have (order 1 in ε and $[|.|]$ is the jump function across ∂S in the direction ν):

$$\begin{cases} \forall \mathbf{v} \in W_0, \\ \\ \alpha \displaystyle\int_{\partial S} \mathbf{u}^2.\mathbf{v} + \beta \int_S \sigma(\mathbf{u}^2).\gamma(\mathbf{v}) = - \int_\Omega \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) = \int_{\partial S} [|(\sigma(\mathbf{u}^1)|].\nu.\mathbf{v}. \end{cases} \quad (38)$$

The restriction of $\mathbf{u}^2$ to Ω_S is denoted by $\mathbf{u}^{2p}$. The existence and uniqueness of this term $\mathbf{u}^{2p} \in [H^1(S)]^2$, div($\mathbf{u}^{2p}$)= 0 on S, are obtained exactly as we did for $\mathbf{u}^{1p}$. Then the

computation of $\mathbf{u}^2$ on Ω_S is also handled as we did for $\mathbf{u}^1$. Therefore $\mathbf{u}^2$ is now defined on the whole domain Ω and is an element of the space W_0. Here again, no additional regularity is required for the computation of $\mathbf{u}^2$ but we shall have a different conclusion for $\beta = 0$.

Remark 10. If $\beta = \gamma = 0$ and $\alpha > 0$ the term $\mathbf{u}^1$ should be solution of (terms of order zero in the assumed asymptotic expansion in ε):

$$\forall \mathbf{v} \in W_0, \ \int_\Omega \sigma(\mathbf{u}^0).\gamma(\mathbf{v}) + \alpha \int_{\partial S} \mathbf{u}^1.\mathbf{v} = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}. \tag{39}$$

This implies that necessarily one should have (here $\sigma(\mathbf{u}^0).\nu$ is the value from Ω_S and it is zero for the contribution from S because $\mathbf{u}^0$ is a rigid body velocity on S):

$$\mathbf{u}^1 = -\sigma(\mathbf{u}^0).\nu \ \text{ on } \partial S \ (\nu \text{ inward } S). \tag{40}$$

It is necessary to prove that the normal component of the stress field $\sigma(\mathbf{u}^0).\nu$ is more regular than the classical result (see for instance J.L. Lions-E. Magenes [31]) which states that it belongs to the space $[H^{-1/2}(\partial S)]^2$. In fact a hidden regularity result that one can derive from the J. Hadamard domain derivation method which has been widely used in mathematical control analysis [30], enables one to prove that $\sigma(\mathbf{u}^0).\nu \in [L^2(\partial S)]^2$. But this is insufficient because $\mathbf{u}^1$ should belong to the space $[H^{1/2}(\partial S)]^2$ in order to have $\mathbf{u}^1$ in the space $[H^1(S)]^2$. Therefore one can point out a weakness of this case which is the lost of some regularity which will have some nasty consequences on the convergence of the penalty method when $\varepsilon \to 0$. This is the case for instance if the boundary of the structure has sharp corners (see P. Grisvard for details [24]). But the penalty-duality method will overcome this difficulty and could be seen as an improvement of this penalty method when $\alpha > 0$ and $\beta = \gamma = 0$ as we show in the following.

Let us now turn to the characterization of $\mathbf{u}^1$ inside Ω_S and S (always with $\alpha > 0$ and $\beta = \gamma = 0$). Hence from the equation at the order one in ε, one obtains:

$$\forall \mathbf{v} \in W_0, \ \int_\Omega \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) + \alpha \int_{\partial S} \mathbf{u}^2.\mathbf{v} = 0 \ \text{ and } \mathbf{u}^{1p} = -\sigma(\mathbf{u}^0).\nu \ \text{ on } \partial S. \tag{41}$$

Here again, as in Remark 9, the system (39) can be split into two independent problems. which characterizes $\mathbf{u}^1$ separately on Ω_S and on S.

$$\begin{cases} \text{on } \Omega_S : \\ \\ \forall \mathbf{v} \in V_0, \ \int_{\Omega_S} \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) = 0, \ \ \mathbf{u}^1 = \mathbf{u}^{1p} \text{ on } \partial S, \\ \\ \text{on } S : \\ \\ \forall \mathbf{v} \in [H_0^1(S)]^2, \ \int_S \sigma(\mathbf{u}^1).\gamma(\mathbf{v}) = 0 \ \text{ and } \mathbf{u}^1 = \mathbf{u}^{1p} \text{ on } \partial S. \end{cases} \tag{42}$$

These two non-homogeneous boundary values problems (Dirichlet condition on ∂S) have each one a unique solution because of the classical properties of the linear elasticity operator. But in this case there are two important features which should be emphasized.

1. The regularity of the boundary of the immersed structure (∂S) is required in order to ensure that $\sigma(\mathbf{u}^0).\nu \in [H^{1/2}(\partial S)]^2$. If it isn't one could make use singular functions due to the corner on ∂S but this is beyond our scope in this paper.

2. The solution $\mathbf{u}^1$ on S is no more a rigid body velocity (in general).

Nevertheless this method ($\alpha > 0$ and $\beta = \gamma = 0$) will be the most adapted to the penalty-duality method that we discuss in section 5 as far as one only considers quasi-static modeling. □

3.3 Convergence results (case $\alpha > 0$ $\beta > 0$ and $\gamma = 0$).

We introduce the error term:

$$\underline{\mathbf{u}}^\varepsilon = \mathbf{u}^\varepsilon - \mathbf{u}^0 - \varepsilon \mathbf{u}^1 - \varepsilon^2 \mathbf{u}^2 \in W_0. \tag{43}$$

From the definition of the different terms and from the triangular inequality[2] and c denoting the continuity constant of the bilinear form of the elasticity operator:

$$\begin{cases} \forall \xi > 0: \quad \displaystyle\int_\Omega \sigma(\underline{\mathbf{u}}^\varepsilon).\gamma(\underline{\mathbf{u}}^\varepsilon) + \frac{\alpha}{\varepsilon} \int_{\partial S} ||\underline{\mathbf{u}}^\varepsilon||^2 + \frac{\beta}{\varepsilon} \int_S \sigma(\underline{\mathbf{u}}^\varepsilon).\gamma(\underline{\mathbf{u}}^\varepsilon) = \\ \\ -\varepsilon^2 \displaystyle\int_\Omega \sigma(\mathbf{u}^2).\gamma(\underline{\mathbf{u}}^\varepsilon) \leq c \left\{ \frac{\xi}{2} \varepsilon^4 ||\mathbf{u}^2||^2_{1,\Omega} + \frac{1}{2\xi} ||\underline{\mathbf{u}}^\varepsilon||^2_{1,\Omega} \right\}. \end{cases} \tag{44}$$

Finally, from classical coerciveness properties deduced from Korn inequality [18], one has the following error estimates where c_0 and c_1 are two positive constants independent on ε (but dependent on the data of the problem):

$$c_0||\underline{\mathbf{u}}^\varepsilon||^2_{1,\Omega} + \frac{\alpha}{2\varepsilon}||\underline{\mathbf{u}}^\varepsilon||^2_{0,\partial S} + \frac{\beta}{2\varepsilon}||\underline{\mathbf{u}}^\varepsilon||^2_{1,S} \leq c_1\varepsilon^4. \tag{45}$$

Let us summarize the previous results in the following statement.

Theorem 3.2. *Let $\mathbf{u}^\varepsilon$ be the solution of (10) and $\mathbf{u}^0, \mathbf{u}^1$ the terms characterized in the asymptotic expansion at (35) and (37). We also assume in this Theorem that $\alpha > 0$ and $\beta > 0$. Then there exists a constant -say $c_2 > 0$- independent on ε and such that:*

$$\begin{cases} ||\mathbf{u}^\varepsilon - \mathbf{u}^0 - \varepsilon\mathbf{u}^1||_{1,\Omega_S} \leq c_2\varepsilon^2, \\ \\ ||\mathbf{u}^\varepsilon - \mathbf{u}^0 - \varepsilon\mathbf{u}^1||_{0,\partial S} \leq c_2\varepsilon^{5/2}, \quad ||\mathbf{u}^\varepsilon - \mathbf{u}^0 - \varepsilon\mathbf{u}^1||_{1,S} \leq c_2\varepsilon^{5/2}. \end{cases} \tag{46}$$

Furthermore the term $\mathbf{u}^0$ is exactly the solution of the model (3) we started from where the displacement of the structure is the prescribed to the boundary ∂S of the fluid.

Corollary 1. *From Theorem 3.2 and making use of the triangular inequality, one obtains the following error bounds between $\mathbf{u}^\varepsilon$ and $\mathbf{u}^0$ where c_3 is a constant independent on ε:*

$$\begin{cases} ||\mathbf{u}^\varepsilon - \mathbf{u}^0||_{1,\Omega_S} \leq c_3\varepsilon, \\ \\ ||\mathbf{u}^\varepsilon - \mathbf{u}^0||_{0,\partial S} \leq c_3\varepsilon, \quad ||\mathbf{u}^\varepsilon - \mathbf{u}^0||_{1,S} \leq c_3\varepsilon. \end{cases} \tag{47}$$

Remark 11. In the eventuality where $\alpha = 0$ (no penalty term on the boundary of S) the similar results as those of Theorem 3.2 are still true but $\mathbf{u}^0$ is the not the solution of the initial problem. Even if it is a rigid body velocity on S, it is not the one of the structure deduced from a correct writing of the principle of the mechanics. For instance the mass of the structure is not appearing in the model. Furthermore we restrict the analysis to quasi-static fluid-structure interaction (small reduce frequencies). Hence this possibility should be forgotten. □

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[2]$\forall a, b \in \mathbb{R}$, $\xi \in \mathbb{R}^+$ one has $ab \leq \frac{2\xi}{a}^2 + \frac{1}{2\xi}b^2$

3.4 Convergence result for $\beta = \gamma = 0$ and $\alpha > 0$.

The results obtained in Theorem 3.2 are still true up to few modifications implying an additional regularity assumption on the term $\mathbf{u}^0$. We just give the statement without details because of the narrow similarity with the one of Theorem 3.2. Similarly to the error bound derived for $\alpha > 0$ and $\beta > 0$, and as far as we just upper bound the error between $\mathbf{u}^\varepsilon$ and $\mathbf{u}^0$, it is not necessary to imply the term $\mathbf{u}^2$. At the opposite of the case $\beta > 0$, it would require also an additional regularity on $\sigma(\mathbf{u}^1).\nu$ and therefore on $\mathbf{u}^0$ too.

Theorem 3.3. *The notations are those introduced in the previous subsection but $\mathbf{u}^0$ is now solution of (24). Furthermore, it is assumed that the solution $\mathbf{u}^0$ of the equations (24) satisfies $\sigma(\mathbf{u}^0).\nu \in [H^{1/2}(\partial S)]^2$. Then there exists a constant $c_4 > 0$ independent on ε such that:*

$$\begin{cases} ||\mathbf{u}^\varepsilon - \mathbf{u}^0||_{1,\Omega} \leq c_4\varepsilon, \\[2mm] ||\mathbf{u}^\varepsilon - u_0||_{0,\partial S} \leq c_4\varepsilon. \end{cases} \tag{48}$$

Remark 12. One should notice that the convergence on S to the rigid body velocity is in ε as in the case where $\beta > 0$. The correctors $\mathbf{u}^1$ have in fact no real interest because the term that we need to estimate is $\mathbf{u}^0$. Let us also emphasize on the interest of this strategy which will clearly appear in the penalty-duality method. $\qquad\square$

4   A first comparison between several penalty methods

In order to compare the different penalty terms we discuss a very simple example which shows the various behavior of the three penalty terms considered in this paper.

4.1   The test case

The model is the following one:

$$\begin{cases} V = \{v \in H^1(]0, L[), \ v(0) = 0\}, \ \text{find } u \in V \ \text{that: } \forall v \in V, \\[3mm] \displaystyle\int_0^L \frac{du}{dx}\frac{dv}{dx} + \frac{\alpha}{\varepsilon}\left(u\left(\frac{L}{2}\right) - u_0\right)v\left(\frac{L}{2}\right) + \int_0^L \left\{\frac{\beta}{\varepsilon}\frac{du}{dx}\frac{dv}{dx} + \frac{\gamma}{\varepsilon}\left(u - u_0\right)v\right\} = 0. \end{cases} \tag{49}$$

The term u_0 is a given constant and ε is the penalty parameter in order to prescribe approximately $u = u_0$ on $]L/2, L[$. The three parameters α, β and γ are chosen equal to 0 or 1 in order to compare their efficiency in the penalty strategy. The solution method used is a first degree finite element method (1000 points). We have plotted the results obtained for two choices of the penalty parameter ε on Figures 2 and 3. The space derivative of the solution is plotted on Figure 4 for $\varepsilon = 10^{-2}$

4.2   Comments on the results

First of all it is worth noting that the method with $\alpha = \gamma = 0$ and $\beta > 0$ is not adapted to prescribe a given movement of the structure. As it is explained in remark 11, the solution is constant on $]L/2, L[$ but it is not the one that one hopes. Nevertheless, if we consider the two terms $\alpha > 0$ and $\beta \geq 0$ or $\beta > 0$ and $\gamma > 0$ the results are are much better. But in the

second case the converge is not as good as than in the first one. If one restricts the penalty term to $\alpha = \beta = 0$ and $\gamma > 0$ the convergence occurs but is slower than in the case where $\alpha > 0$, even if this is the only term which is kept.

Concerning the derivatives of the solution with respect to the space coordinate one can see on Figure 4 that the convergence is not satisfying if $\alpha = 0$. This the reason why we develop in the next section 5 a penalty-duality method which implies only this boundary term. Furthermore, the dual variable (Lagrange multiplier) is easier to handle when only this boundary term is taken into account. In case of internal conditions the duality is a little bit more complex. We refer to the so-called Arlequin method developed by H. Ben Dhia [6]-[7] and in slightly different context by T. Belytschko & all [3].

5   The penalty-duality method

In this section, we consider the case where $\alpha > 0$ and $\beta = \gamma = 0$. Because the penalty model leads to ill conditioning, one can suggest to use a variant of this strategy introduced fifty years ago by D. Bertsekas [9] and known as the penalty-duality method. The point is to satisfy exactly the constraints which are considered in the penalty term by a dual treatment. Hence, for any $r > 0$ (r stands for $1/\varepsilon$ but will be moderate at the opposite of what happens in the penalty strategy), we introduce the Lagrangian defined by:

$$L(\mathbf{v}, \lambda, \mu) = \frac{1}{2} \int_\Omega \sigma(\mathbf{v}).\gamma(\mathbf{v}) + \frac{\alpha r}{2} \int_{\partial S} ||\mathbf{v} - \mathbf{u}_0||^2 + \int_{\partial S} \mu.(\mathbf{v} - \mathbf{u}_0) - \int_{\Gamma_1} \mathbf{g}.\mathbf{v}, \quad (50)$$

where $\mathbf{v} \in W_0$, $\mu \in [H^{-1/2}(\partial S)]^2$ (we use the notation with the integral for the duality[3] but we know that it is an abuse; see remark 13). One can add the penalty term inside S (see [4]-[5]-[29]):

$$\frac{\gamma}{\varepsilon} \int_S |\mathbf{u} - \mathbf{u}_0|^2.$$

It is not really useful for the static case that we consider here but it is a necessity for the dynamic case as those treated in the numerical tests of subsections 8.2 and 8.3.

A saddle point of L is an element $(\mathbf{u}^r, \lambda) \in W_0 \times [H^{-1/2}(\partial S)]^2$ such that:

$$\begin{cases} \forall \mathbf{v} \in W_0, \ \int_\Omega \sigma(\mathbf{u}^r).\gamma(\mathbf{v}) + \alpha r \int_{\partial S} (\mathbf{u}^r - \mathbf{u}_0).\mathbf{v} + \int_{\partial S} \lambda.\mathbf{v} = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}, \\ \\ \forall \mu \in [H^{-1/2}(\partial S)]^2, \ \int_{\partial S} \mu.(\mathbf{u}^r - \mathbf{u}_0) = 0, \end{cases} \quad (51)$$

Remark 13. The writing with an integral of the duality between the spaces $[H^{-1/2}(\partial S)]^2$ and $[H^{-1/2}(\partial S)]^2$ is a familiarity which is not correct because the functions involved are not in the space $L^2(\partial S)$. It just a commodity (in our mind). □

The existence and uniqueness are relevant of the general A.K. Aziz and I. Babuska Theorem [2] that we recall hereafter for sake of convenience.

Theorem 5.1. *[Aziz-Babuska] Let H_1 and H_2 be two Hilbert spaces, b a bilinear and continuous form on $H_1 \times H_2$ and j a linear and continuous form on H_2. It is asuumed that the following properties hold:*

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[3]the dual of $H^{1/2}(\partial S)$ is $H^{-1/2}(\partial S)$ in $1D$ which is the case here.
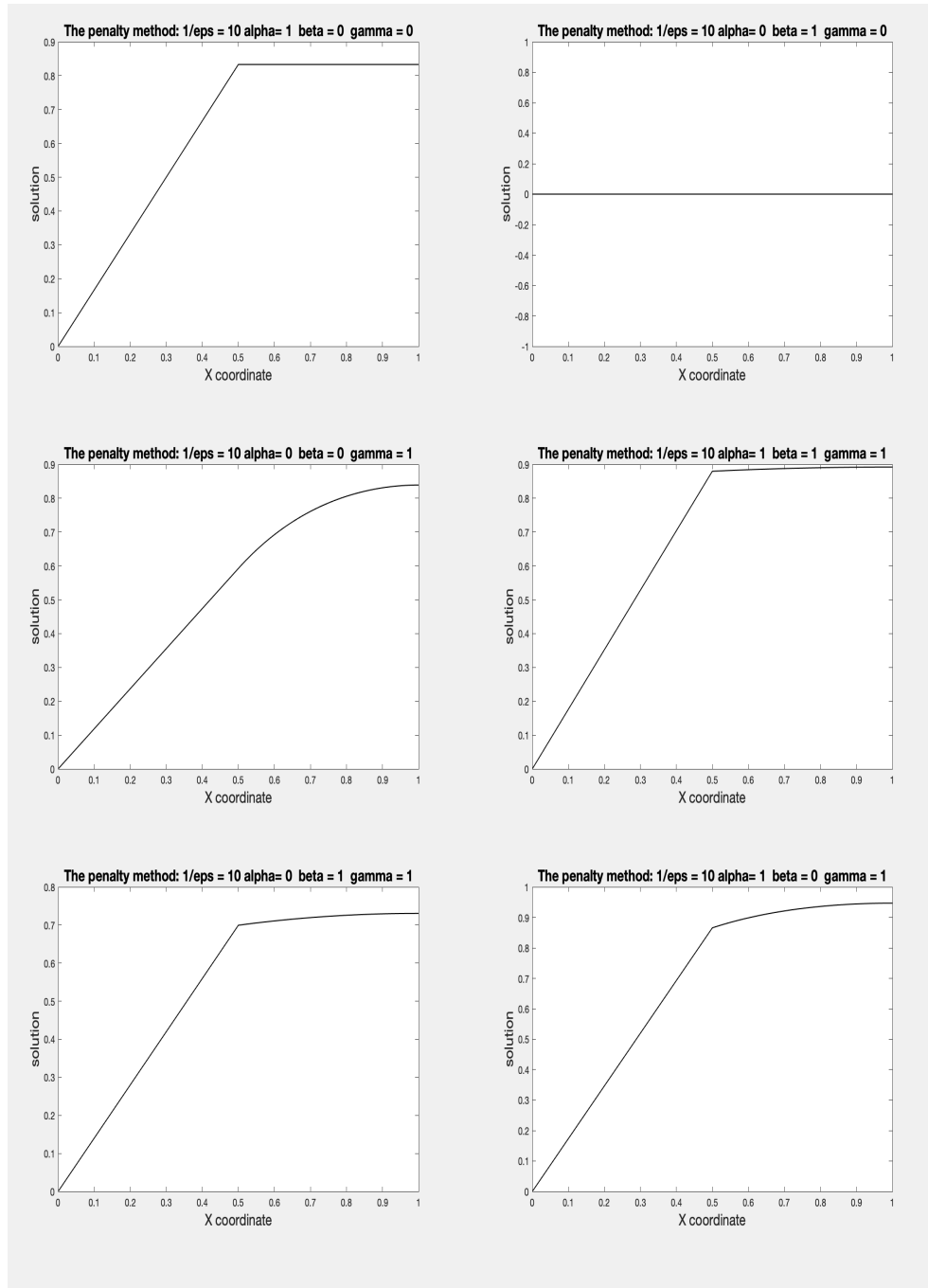
Figure 2: Penalty solution for $\varepsilon = 0.1$

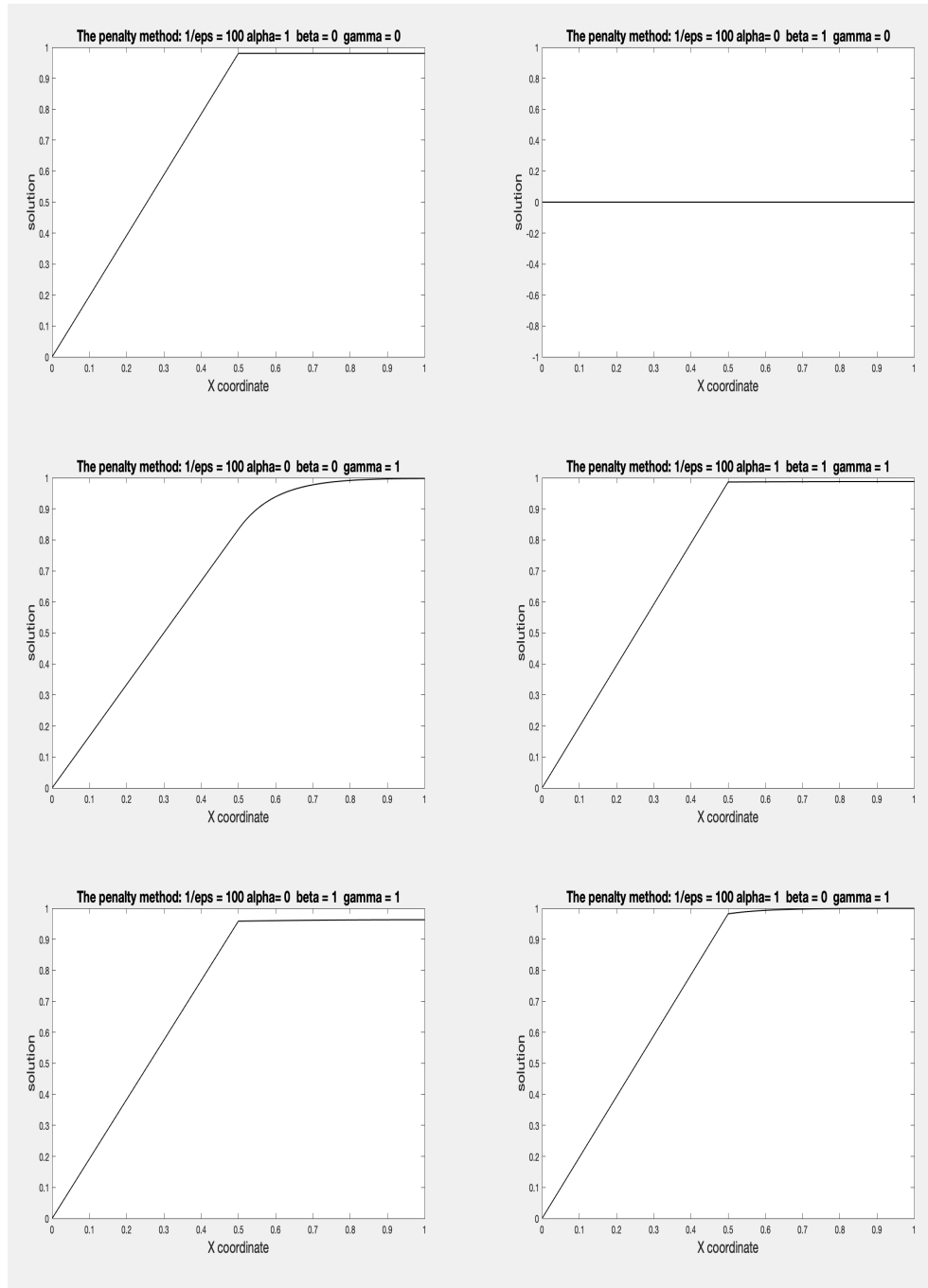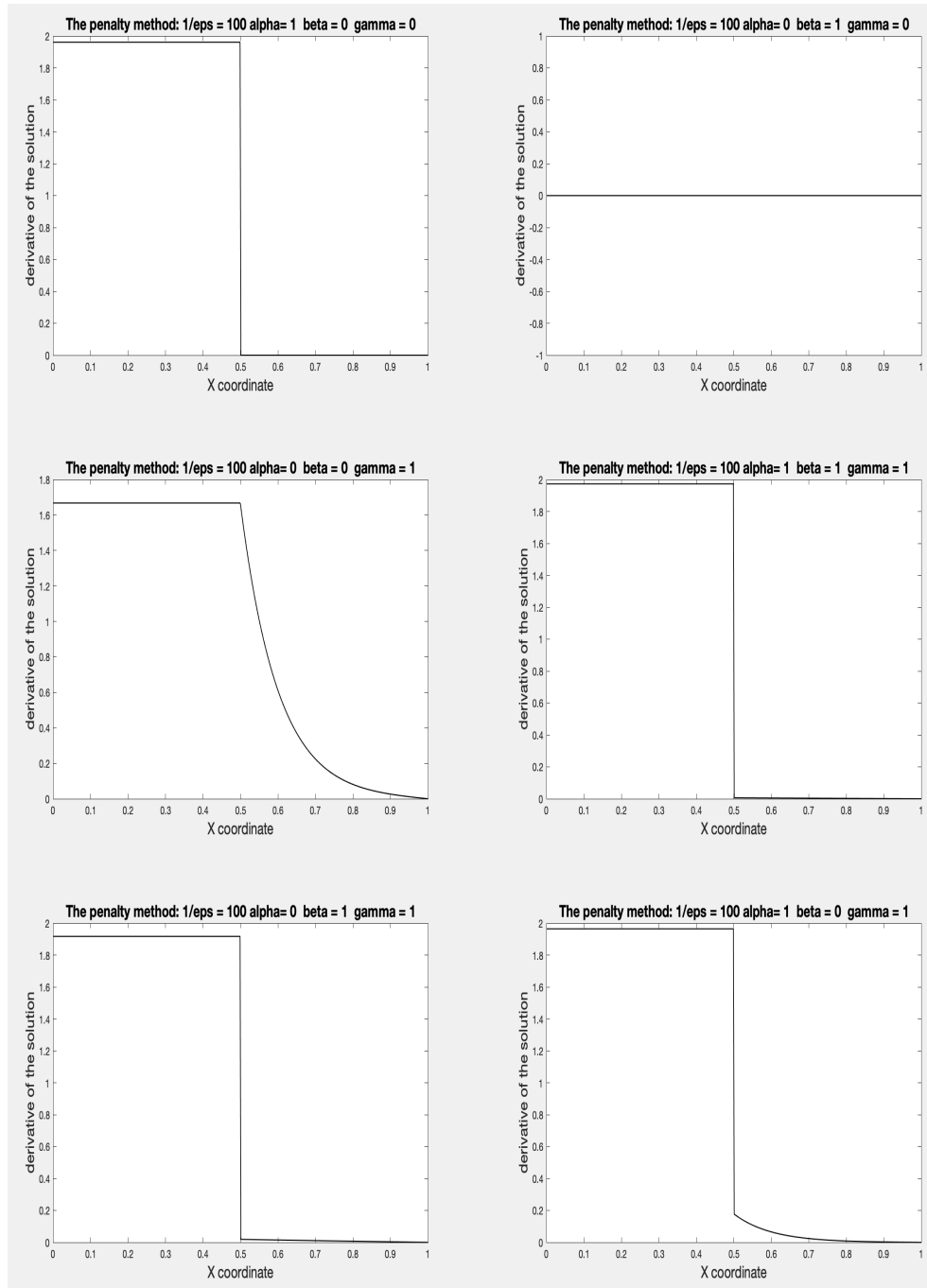Figure 3: Penalty solution for $\varepsilon = 0.01$

Figure 4: Derivative in x of the penalty solution for $\varepsilon = 0.01$

 # 1 *if* $q \in H_2$ *is such that:* $\forall \mu \in H_1, \ b(\mu, q) = 0$ *then* $q = 0$;

 # 2 *there exists a constant* $\delta > 0$ *such that* $\forall \mu \in H_1, \ \displaystyle\sup_{q \in H_2} \frac{b(\mu, q)}{||q||_{H_2}} \geq \delta ||\mu||_{H_1}$.

Then there exists a unique element $\lambda \in H_1$ *such that:*

$$\forall q \in H_2, \ b(\lambda, q) = j(q). \tag{52}$$

Let us now apply this Theorem to our case.

Theorem 5.2. *Existence and uniqueness of a solution to (51).* *Let* $r > 0$ *and* $\alpha > 0$. *The function* **g** *is assumed as previously to be in the space* $[L^2(\Gamma_1)]^2$. *Hence the system (48) has a unique solution* $(\mathbf{u}^r, \lambda)$ *in the space* $W_0 \times \left[H^{-1/2}(\partial S) \right]^2$. *Furthermore* $\mathbf{u}^r = \mathbf{u}$ *solution of the initial problem (3).*

Proof. If $(\mathbf{u}^r, \lambda) \in W_0 \times \left[H^{-1/2}(\partial S) \right]^2$ is a solution it should satisfy the following relation:

$$\forall \mathbf{v} \in W_{00}, \ \int_{\Omega} \sigma(\mathbf{u}^r).\gamma(\mathbf{v}) = \int_{\Gamma_1} \mathbf{g}.\mathbf{v}, \ \text{and necessarily: } \mathbf{u}^r = \mathbf{u}_0 \text{ on } \partial S. \tag{53}$$

Therefore $\mathbf{u}^r$ is the initial solution of equation (3) which is also the term $\mathbf{u}^0$ computed in the asymptotic method(s) applied to the penalty method at section 3 (for both $\alpha > 0, \ \beta = 0$ or $\alpha > 0, \beta > 0$). It also implies that it is unique and equal to the rigid body velocity $\mathbf{u}_0$ on the whole domain S and not only on its boundary ∂S.

Finally, in order to complete the proof of Theorem 5.2 it is sufficient to prove the existence and uniqueness of λ using Aziz-Babuska Theorem 5.1.

Let us set:

$$\begin{cases} H_1 = \left[H^{-1/2}(\partial S) \right]^2, \ H_2 = \left[H^{-1/2}(\partial S) \right]^2, \\[2mm] \forall \mu \in H_1, \ \forall \mathbf{v} \in H_2, \ b(\mu, \mathbf{v}) = \alpha \int_{\partial S} \mu.\mathbf{v}, \\[2mm] \forall \mathbf{v} \in H_2, \ j(\mathbf{v}) = \int_{\Omega_S} \sigma(\mathbf{u}^r).\gamma(\mathbf{v}) - \int_{\Gamma_1} \mathbf{g}.\mathbf{v}. \end{cases} \tag{54}$$

The first requirement of the Aziz-Babuska Theorem is obvious because of the duality between H_1 and H_2. The second one is satisfied with $\delta = 1$ because of the definition of the norm in H_1.

The only point to be verified is the definition and the continuity of j on H_2. First of all j is clearly linear. It depends on the value of $\mathbf{v}$ on the boundary ∂S because of the definition of $\mathbf{u}^r = \mathbf{u}^0$. Hence, for a given element $\mathbf{v}$ in H_2 one can associate any element in the space V_0 denoted also by $\mathbf{v}$ and which gives the same value of j. Let us recall that there exists a linear and continuous operator -say R- from H_2 into V_0 [31] such that:

$$\exists c_9 > 0, \text{ independent on } \mathbf{v} \in H_1 \text{ suh that: } ||R(\mathbf{v})||_{1,\Omega_S} \leq c_9 ||\mathbf{v}||_{1/2,\partial S}. \tag{55}$$

Hence:

$$|j(\mathbf{v})| = |\int_{\Omega_S} \sigma(R\mathbf{v}).\gamma(R\mathbf{v}) - \int_{\Gamma_1} \mathbf{g}.R(\mathbf{v})| \leq c_{10} ||\mathbf{v}||_{1/2,\partial S}, \tag{56}$$

Hence the assumptions of Theorem 5.1 are satisfied and we can conclude the proof of Theorem 5.2. $\qquad\square$

Remark 14. At the opposite of what occurs in the penalty method, the solution of the penalty-duality model is exactly the one of the initial problem. This is due to the fact the duality ensures that the penalty term is zero. This term (the one in factor of r) only plays a role for ensuring a nice and fast convergence of the Uzawa duality algorithm). Therefore it is not necessary to seek for an optimal step -say ϱ- in the gradient of the dual problem: the choice $\varrho = r$ ensures the convergence. This is the advantage of the D. Bertsekas method [9]. □

Remark 15. It is a basic point to notice that the solution $\mathbf{u}^r$ is independent on r. Therefore it is not necessary to choose a large value as in the penalty model (r is the equivalent of $1/\varepsilon$) . Furthermore one could point out that the strategy could also be applied without the penalty term but in this case the algorithm studied in the next section is more difficult to use as we underline in the following (difficulty in adjusting the gradient step in the Uzawa solution method [9], [22]). If one uses a conjugate gradient on the dual problem (in λ), the operator has a better condition number if one add the penalty term. □

Remark 16. A possibility that we do not recommend consists in considering the two penalty terms ($\alpha > 0$ and $\beta > 0$). Nevertheless the dual space for the Lagrange multiplier of the the constraint $\gamma_{ij}(\mathbf{u}) = 0$ on S is a little bit complex and not very convenient in a practical application. Furthermore the numerical implementation seems to be some more complicated. □

6 The numerical algorithm and its solution method

In this section we focus on the Uzawa algorithm [22].[11] for solving (51). Let us start from a given value for $\lambda^{n_0} \in [H^{-1/2}(\partial S)]^2$ (the most classical choice is to start from $\lambda^{n_0} = 0$). For each $n \geq n_0$, we define $\mathbf{u}^n \in W_0$ solution of:

$$\forall \mathbf{v} \in W_0, \ \int_\Omega \sigma(\mathbf{u}^n).\gamma(\mathbf{v}) + r \int_{\partial S} (\mathbf{u}^n - \mathbf{u}_0).\mathbf{v} = \int_{\Gamma_1} \mathbf{g}.\mathbf{v} - \int_{\partial S} \lambda^n.\mathbf{v}. \quad (57)$$

Then we upgrade λ^n by setting:

$$\lambda^{n+1} = \lambda^n + r(\mathbf{u}^n - \mathbf{u}_0) \ \text{on} \ \partial S. \quad (58)$$

An important point is that if the first multiplier λ^n belongs to the space $[L^2(\partial S)]^2$ all the sequence λ^n belongs to this space. But in general the solution λ of (51) doesn't belongs to this space unless one has a regularity property. In fact, one has (interpretation of (51)), $\lambda = -\sigma(\mathbf{u}^r).\nu$ on ∂S. This regularity assumption is exactly the one used in the asymptotic analysis of the penalty model for $\alpha > 0$ and $\beta = 0$. It is convenient (but not necessary) in the following to adopt this hypothesis (the norm $L^2(\partial S)$ is more convenient in the writings than the one of $H^{-1/2}(\partial S)$).

The convergence of the algorithm is classical [9]. But for sake of clarity let us summarize the proof hereafter with few remarks concerning the regularity of the multiplier λ which are only valid for our case.

Theorem 6.1. *Let* $(\mathbf{u}^n, \lambda^n) \in W_0 \times [H^{-1/2}(\partial S)]^2$ *be the sequence defined by the algorithm (57)-(58). We assume (this is satisfied as soon as there is no singularities due to sharp corners on ∂S), just for sake of convenience in the writings, that:*

$$\lambda = -\sigma(\mathbf{u}^r).\nu \in [L^2(\partial S)]^2.$$

Then for any $r > 0$:

$$\begin{cases} \lim_{n \to \infty} \mathbf{u}^n = \mathbf{u}^r = \mathbf{u} \text{ solution of (3)}, \\ \lim_{n \to \infty} \lambda^n = \lambda \text{ solution with } \mathbf{u}^r \text{ of (51)}. \end{cases} \tag{59}$$

Proof. Let us introduce the *gap* variables:

$$\underline{\mathbf{u}}^n = \mathbf{u}^n - \mathbf{u}^r, \quad \underline{\lambda}^n = \lambda^n - \lambda.$$

One has the following equalities:

$$\begin{cases} \forall \mathbf{v} \in W_0, \int_\Omega \sigma(\underline{\mathbf{u}}^n).\gamma(\underline{\mathbf{u}}^n) + r \int_{\partial S} |\underline{\mathbf{u}}^n|^2 + \int_{\partial S} \underline{\mathbf{u}}^n \underline{\lambda}^n = 0, \\ |\underline{\lambda}^{n+1}|^2 = |\underline{\lambda}^n|^2 + r^2 |\underline{\mathbf{u}}^n|^2 + 2r\underline{\lambda}^n.\underline{\mathbf{u}}^n. \end{cases}$$

We deduce that:

$$\int_{\partial S} |\underline{\lambda}^n|^2 - \int_{\partial S} |\underline{\lambda}^{n+1}|^2 = r[-2 \int_{\partial S} \underline{\mathbf{u}}^n \underline{\lambda}^n - r \int_{\partial S} |\underline{\mathbf{u}}^n|^2],$$

or else:

$$\int_{\partial S} |\underline{\lambda}^n|^2 - \int_{\partial S} |\underline{\lambda}^{n+1}|^2 = r[2 \int_\Omega \sigma(\underline{\mathbf{u}}).\gamma(\underline{\mathbf{u}}^n) + 2r \int_{\partial S} |\underline{\mathbf{u}}^n|^2 - r \int_{\partial S} |\underline{\mathbf{u}}^n|^2]$$

$$= r[2 \int_\Omega \sigma_{ij}(\underline{\mathbf{u}}).\gamma(\underline{\mathbf{u}}^n) + r \int_{\partial S} |\underline{\mathbf{u}}^n|^2] \geq 0.$$

Because the real and positive sequence of numbers $q^n = \int_{\partial S} |\underline{\lambda}^n|^2$ is decreasing, it is convergent and therefore it is a Cauchy sequence. Hence $\lim_{n \to \infty} q^n - q^{n+1} = 0$. As a consequence (using the $H^1(\Omega)$-coerciveness of the bilinear form $\int_\Omega \sigma(\mathbf{u}).\gamma(\mathbf{v})$):

$$\begin{cases} \lim_{n \to \infty} ||\mathbf{u}^n - \mathbf{u}_0||_{0,\partial S} = 0, \\ \lim_{n \to \infty} ||\mathbf{u}^n - \mathbf{u}^r||_{1,\Omega} = 0. \end{cases}$$

The second convergence result reinforces the first one because of the continuity of the trace mapping from $H^1(\Omega)$ into $H^{1/2}(\partial S)$:

$$\lim_{n \to \infty} ||\mathbf{u}^n - \mathbf{u}_0||_{1/2,\partial S} = 0.$$

The final step consists in proving the convergence of the multiplier λ^n. From:

$$\int_{\partial S} \underline{\lambda}^n.\mathbf{v} = - \int_\Omega \sigma(\underline{\mathbf{u}}^n).\gamma(\mathbf{v}) - r \int_{\partial S} \underline{\mathbf{u}}^n.\mathbf{v} - \int_{\Gamma_1} \mathbf{g}.\mathbf{v},$$

and because of the property # 2 mentioned in Theorem 5.1, one has:

$$\lim_{n \to \infty} ||\lambda^n - \lambda||_{-1/2,\partial S} = 0.$$

Even if we assumed the regularity $\lambda \in [L^2(\partial S)]^2$ the standard convergence is only in the space $[H^{-1/2}(\partial S)]^2$. □

Remark 17. The main advantage in the choice of the penalty term reduced on the boundary of the structure is that it enables to decouple completely the numerical software and just requires a geometrical prolongation mapping from the boundary ∂S of the structure into the open set Ω and a geometrical restriction mapping from Ω on to ∂S. This advantage was underlined by many authors and a nice presentation by J. Hovnanian is given in [26]. $\square$

7 Few elementary tests for the penalty-duality strategy

Just in order to make a link with the elementary test introduced in section 4, we apply the penalty-duality algorithm to the same model. The results are plotted on Figures 5 and 6. They show the efficiency of this method regarding the precision even for ε not so small. The numerical implementation in a complex model is certainly another discussion which will be carried out in A. Falaise and E. Liberge [19]. In particular, the mapping connecting the boundary ∂S with the global mesh is a cornerstone problem regarding the multiprocessor programming.
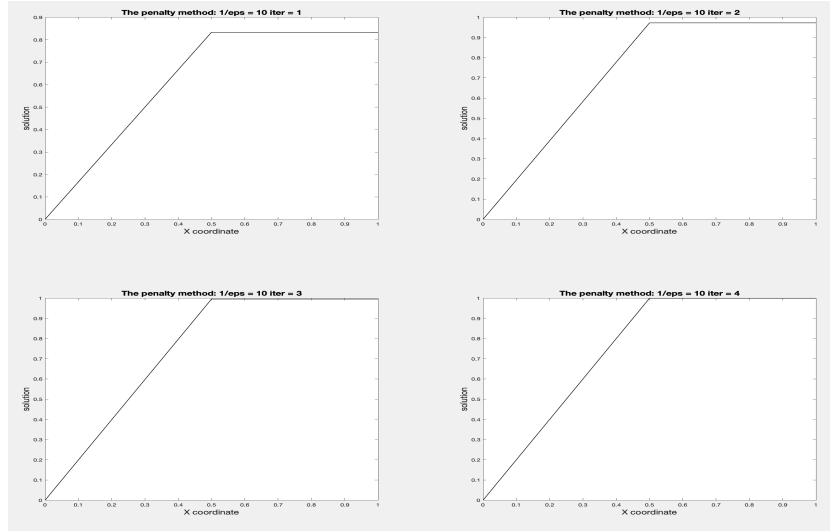
Figure 5: Solution obtained with the penalty-duality strategy with $\varepsilon = .1$

8 Application to convection-diffusion equation

We consider two examples: the first one is the linear advection-diffusion model and the second one is the Burgers equation. Nevertheless it is worth to point out that the boundary of the structure are not updated..
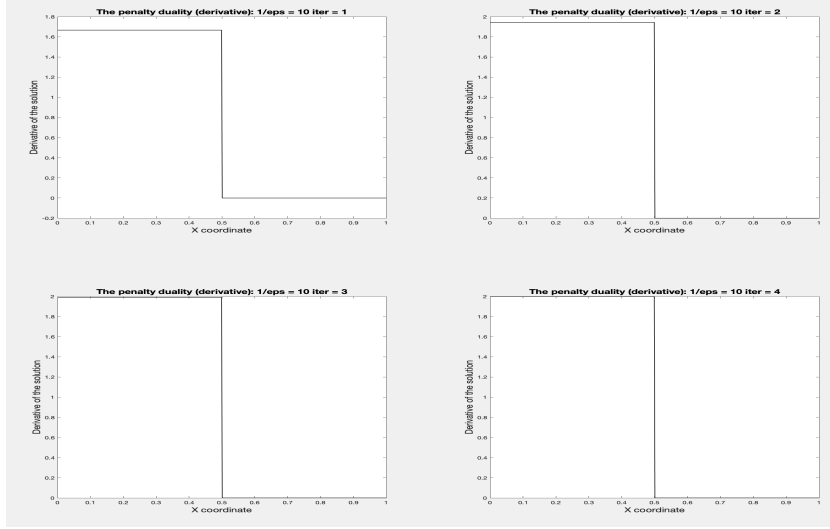
Figure 6: Derivative of the solution obtained with the penalty-duality strategy with $\varepsilon = .1$ during the four first iterations

8.1 The linear advection diffusion model

We consider the following $1D$ advection-diffusion equation where $u \in \mathcal{C}^0([0,T] \times [0,L])$ is the unknown (where $c > 0$ $u_0(t)$ and $d(x)$ are given):

$$
\begin{cases}
x \in]0,a[\cup]b,L[, \ t \geq 0 : \ \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} - \nu\dfrac{\partial^2 u}{\partial x^2} = 0, \\[2mm]
t \geq 0 \ u(0,t) = u(L,t), \ \ x \in]0,a[\cup]b,L[: \ u(x,0) = u_0(x), \\[2mm]
t \geq 0, \ x \in]a,b[: \ u(x,t) = d_0(t) \ (\text{ sine function}).
\end{cases}
\tag{60}
$$

We denote by Δx (respectively Δt) the space step (respectively the time step). The lower index indicate the space discretization and the upper one the time discretization. The numerical scheme that we consider for this equation in $(]0,a[\cup]b,L[) \times]0,T[$ leads to:

$$
\begin{cases}
\forall i \in \{2,i_a\} \cup \{i_b, N-1\}, \\[2mm]
\dfrac{u_i^{n+1} - u_i^n}{\Delta t} + c\left(\dfrac{u_i^n - u_{i-1}^n}{\Delta x}\right) - \nu\left(\dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right) = 0, \\[2mm]
u_i^0 = u_0(i\Delta x) \ \text{initial conditions}, \\[2mm]
\forall n \geq 0 : \ u_1^n = u_N^n, \ \text{boundary conditions}, \\[2mm]
u_{i_a}^n = u_{i_b}^n = d_0^n \ \text{boundary of the structure}, \\[2mm]
\text{where } i_a \text{ and } i_b \ \text{are the indices corresponding to } a \text{ and } b.
\end{cases}
\tag{61}
$$

The penalty-duality formulation is the following one (δ_{i,i_c} is the Kronecker symbol $= 1$ if $i = i_c$ and 0 else and we introduce a penalty term with the parameter γ as before):

$$
\begin{cases}
\forall i \in \{2, N-1\}, \\[2mm]
\dfrac{u_i^{n+1} - u_i^n}{\Delta t} + \dfrac{\gamma}{\varepsilon}(u_i^{n+1} - d_0^{n+1}) + c(\dfrac{u_i^n - u_{i-1}^n}{\Delta x}) + r[(u_i^{n+1} - d_0^{n+1})\delta_{i,i_a} + \\[2mm]
(u_i^{n+1} - d_0^{n+1})\delta_{i,i_b}] - \nu(\dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}) + \\[2mm]
\lambda^{n+1}(i_a)\delta_{i,i_a} + \lambda^{n+1}(i_b)\delta_{i,i_b} = 0, \quad u_i^0 = u_0(i\Delta x) \text{ initial conditions,} \\[2mm]
\forall n \geq 0: \ u_1^n = u_N^n \text{ and } u_{-1}^n = u_{N-1}^n \text{ periodic boundary conditions,} \\[2mm]
\forall n \geq 0: \ u_{i_a}^n = u_{i_b}^n = d_0^n \text{ boundary of the structure.}
\end{cases}
\tag{62}
$$

For each n the duality algorithm (the iterations are indexed by p) is defined as follows:

\# $p = 0$, $\lambda_{i_a}^0 = \lambda_{i_b}^0 = 0$;

\# compute u_i^{n+1} solution of (62) $\lambda^p = (\lambda_{i_a}^p, \lambda_{i_b}^p)$ being fixed;

\# set: $\lambda^{p+1} = \lambda^p + r(u_{i_a}^n - d_0^n, u_{i_b}^n - d_0^n)$

\# convergence test => stop or $p = p + 1$

8.2   The numerical tests for the advection-diffusion (linear)

We have tested the two possibilities: a) no duality and b) with duality. The results are plotted on Figures 7(a) and 7(b) for the first case and on Figures 8(a) and 8(b) for the case with duality. We chose the following data set:

\# the kinematics viscosity is $\nu = 0.001$;

\# the coefficient of the Bertsekas algorithm is $r = 10$;

\# the length of the space interval is $L = 1$ and the length of the structure is $L/10$ and the coordinates of the two extremities are denoted by $x_a = .45L$, $x_b = .55L$;

\# the time dependence of the prescribed velocity of the structure is $u_{st}(t) = \sin(2\pi t)$ and $c = 1$ for the advection-diffusion case. The space dependence in the structure is (just for an example):
$$u_{sx}(x) = 0.4 + 2(2x - x_a - x_b)/L.$$
Hence the velocity prescribed in the structure is $u_{st}(t)u_{sx}(x)$;

\# the initial condition outside the structure is (still for example):

$u(x, 0) = 4.4(x_a - x - L/5)$ if $x \in [0, x_a - L/5]$,
$u(x, 0) = 4.4(x - x_b - L/5)$ if $x \in [x_b + L/5, L]$ and $u(x, 0) = 0$ elsewhere;

\# the penalty parameter used for the periodic conditions at $x = 0$ and $x = L$ but also for the penalty term inside the structure is $\varepsilon = 10^{-3}$;

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\# the time delay is $T = 2$ and there are 1000 time steps and 500 space steps.

There is a meaningful difference between the two cases and mainly on the derivatives with respect to x if $\varepsilon > 10^{-8}$. For $\varepsilon < 10^{-8}$ the penalized solution is quite the one obtained with the penalty-duality method. In other words the duality algorithm is not useful. Nevertheless, for such values of the penalty parameter the solver requires a large enough viscosity which is not so drastic for larger values of ε. Furthermore the one dimensional test aren't meaningful concerning the condition number of two or three dimensional models. Our goal in this paper is only to discuss the advantages and the drawbacks of this added duality algorithm for fluid-structure models from a theoretical point of view. Figures 7 show the solution (ad) (Figure 7(a)) and the derivative versus x of the solution (ad) (Figure 7(b)) obtained without neither penalty inside the structure nor duality, but with a penalty term on the boundary between the structure and the fluid. The structure is moving at a given frequency (2 periods have been plotted). This induces waves in the fluid. One should notice on Figure 7(b) that the propagation of stress waves is very damped in the fluid. The solutions obtained with the duality method are plotted on Figure 8. The normal stress at the two boundaries versus time, with and without duality are plotted Figure 9.

(a) Solution   (b) The derivative versus x of the solution (ad)
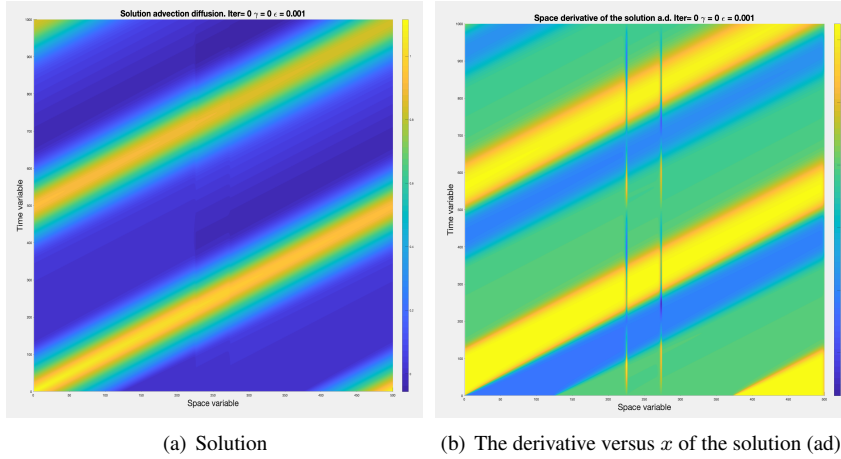
Figure 7: Solution (ad) obtained without neither penalty inside the structure nor duality. But there is a penalty term on the boundary between the structure and the fluid.

Next, the penalty term inside the structure is added. Result with and without duality are plotted on Figures 10 and 11. The duality algorithm does not affect the normal stress at the fluid-solid interface as far as ε is very small (Figure 12) but the use of the penalty term makes an important difference.

8.3 A non linear case (Burgers)

In this subsection we consider the following non linear model (Burgers equation) but the boundary of the structure aren't updated. The scheme used is similar to the one used is the linear case (fully implicit for the diffusion and Godounov scheme for the transport) and the
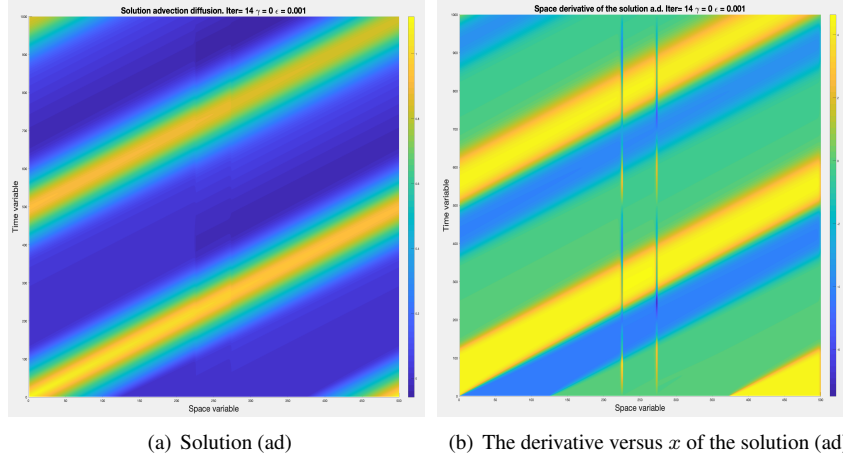
(a) Solution (ad)

(b) The derivative versus x of the solution (ad)

Figure 8: Solution (ad) obtained with the duality method on the boundary of the structure but no penalty term inside the structure.
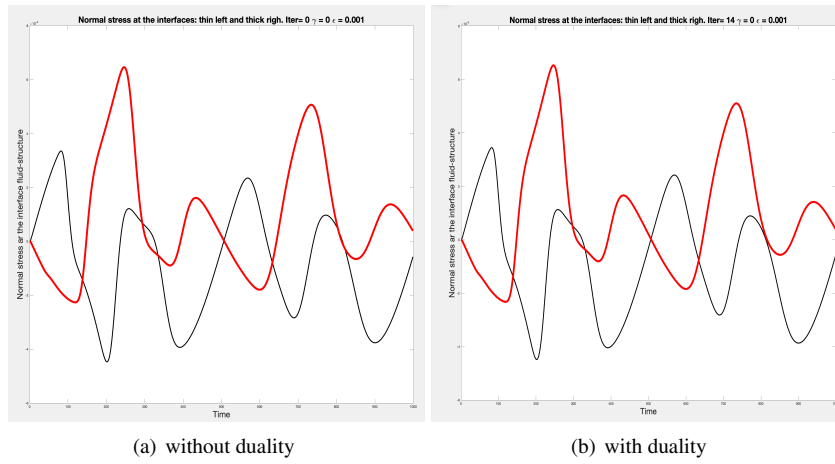

(a) without duality

(b) with duality

Figure 9: The normal stress at the two boundaries of the structure versus the time

(a) Solution

(b) Space derivative of the solution of the advection-convection equation
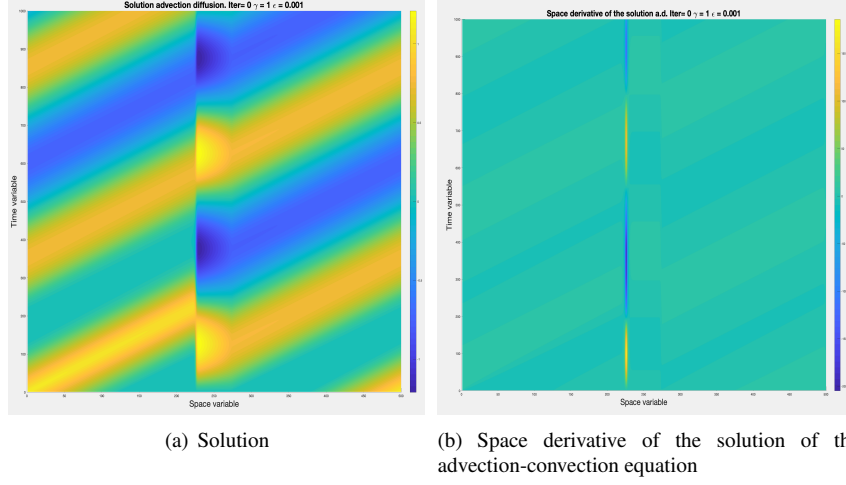
Figure 10: Solution of the advection-convection equation with two penalty terms: one on the boundary between the fluid and the structure and the other inside the structure. But no duality iteration have been used.

(a) Solution

(b) Space derivative of the solution of the advection-convection equation
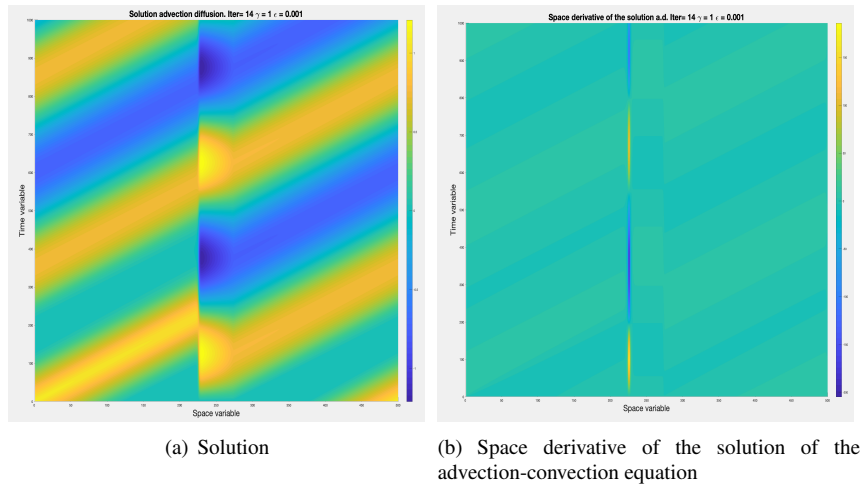
Figure 11: solution of the advection-convection equation with the two penalty terms and the duality algorithm.

(a) no duality   (b) duality
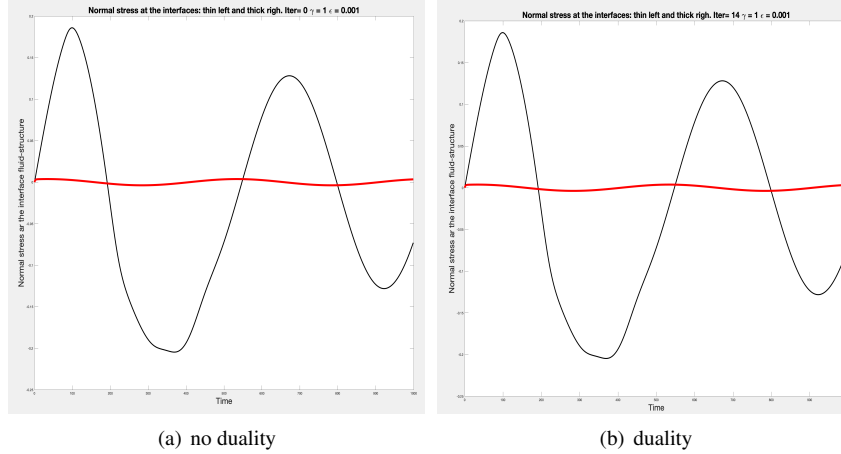
Figure 12: This Figure represents the normal stress at the f-s interfaces; the two penalty terms have been used. The difference with Figure 9 (no penalty inside the structure) can be noticed.

data are also the same concerning the dimensions and the initial values.

$$
\begin{cases}
x \in]0, a[\cup]b, L[, \ t \geq 0 : \ \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(\dfrac{u^2}{2}) - \nu \dfrac{\partial^2 u}{\partial x^2} = 0; \\[2mm]
t \geq 0 \ u(0, t) = u(L, t), \ \ x \in]0, a[\cup]b, L[: \ u(x, 0) = u_0(x) \\[2mm]
t \geq 0, \ x \in]a, b[: \ u(x, t) = d_0(t) \ (\text{ sine function}).
\end{cases}
\tag{63}
$$

8.3.1 The numerical results

First of all let us point out that the results should be compared with the quasi-exact solution obtained with a very small value of the penalty parameter ε but which requires some precautions in the adjustment of the numerical tests (ill conditioning). The quasi-exact solution and its space derivatives are plotted on Figures 21(a), 21(b), 22(a) and 22(b).

The first results have been plotted on Figures 13(a), 13(b) and 14 for the case where there is no penalty term inside the structure and no duality. But there is a moderate penalty term at the f-s interfaces. If we compare to the quasi-exact solution plotted on Figure 21(a) one can see a big difference. The same is true (and even amplified for the space derivatives (see Figures 13(b) and 14 compared to Figures 21(a) and 22(a)).

On the Figures 15(a), 15(b), 16(a) and 16(b) we have plotted the solution of Burgers equation obtained with the penalty-duality algorithm but without penalty term (L^2-norm between the velocity obtained by the scheme and the one prescribed in the structure). The results compared with the quasi-exact solution are much better but not satisfying.

The penalty term inside the structure without duality has been added in the computations shown on Figures 17(a), 17(b) and 18. The results are closer to the quasi-exact one of Figures 21(a), 21(b)-22(a). The improvement with adding the duality algorithm is small for small values of the penalty parameter as one can see on Figures 19(a), 19(b), 20(a) and 20(b) (compared to the quasi-exact solution), but exists.

(a) Solution (Burgers)

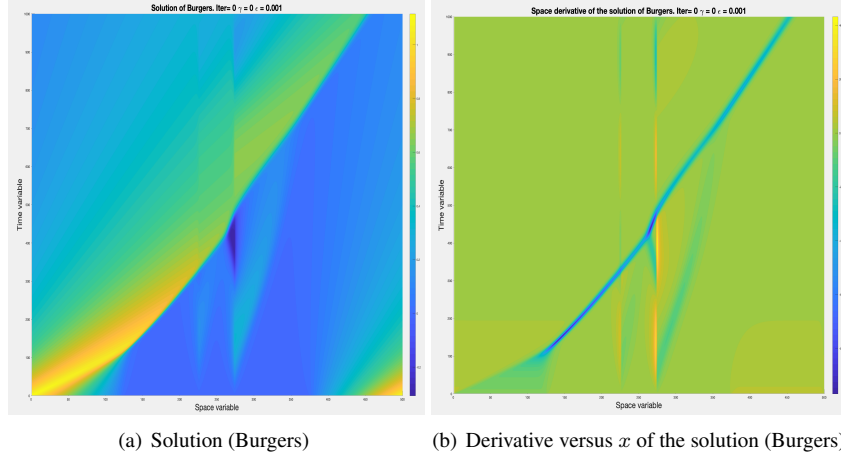(b) Derivative versus x of the solution (Burgers)

Figure 13: Solution (Burgers) without duality and no penalty term inside the structure (same data as for the linear case) but $r = .1$
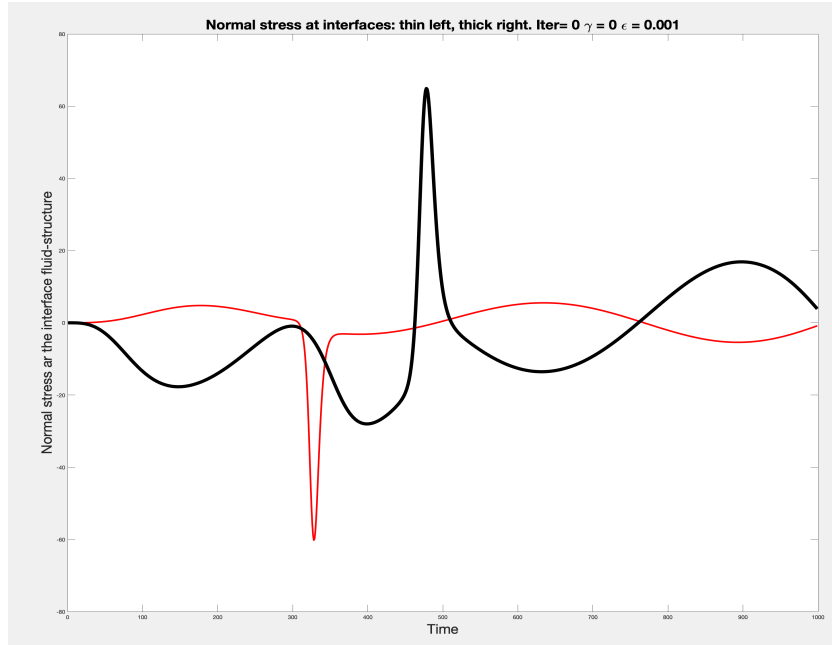

Figure 14: Normal stress at the f-s interface for the Burgers solution without duality and no penalty term inside the structure (same data as for the linear case)
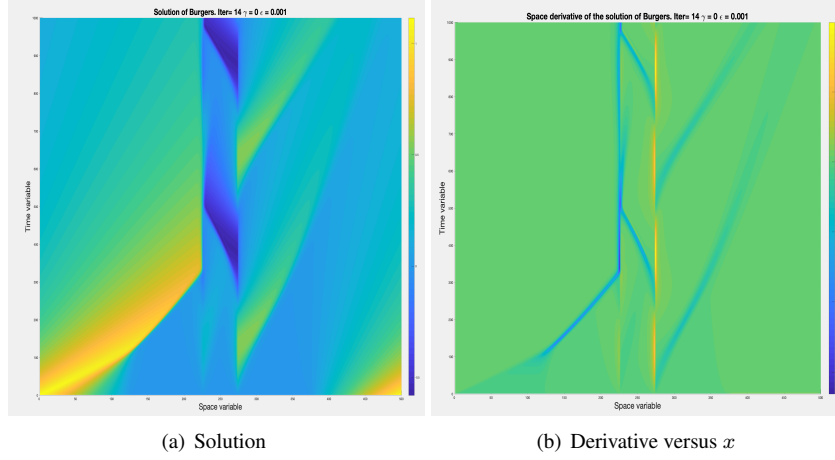
(a) Solution

(b) Derivative versus x

Figure 15: The solution with duality (Burgers) but no penalty term at the f-s interface. The data are still the same as in the linear case. One can see a meaningful difference with the case without duality.

(a) Normal stress at the f-s interface

(b) The Lagrange multipliers at the two f-s interfaces
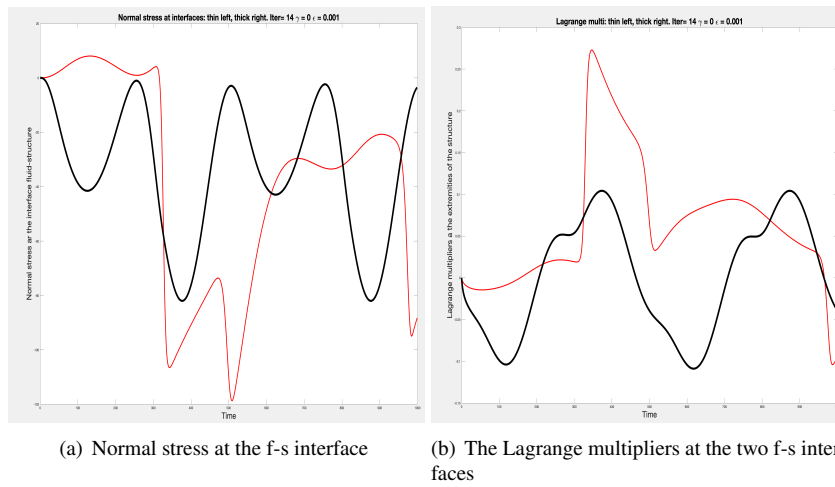
Figure 16: Burgers solution obtained with duality but no penalty term inside the structure.

(a) solution
(b) Derivative versus x of Burgers solution
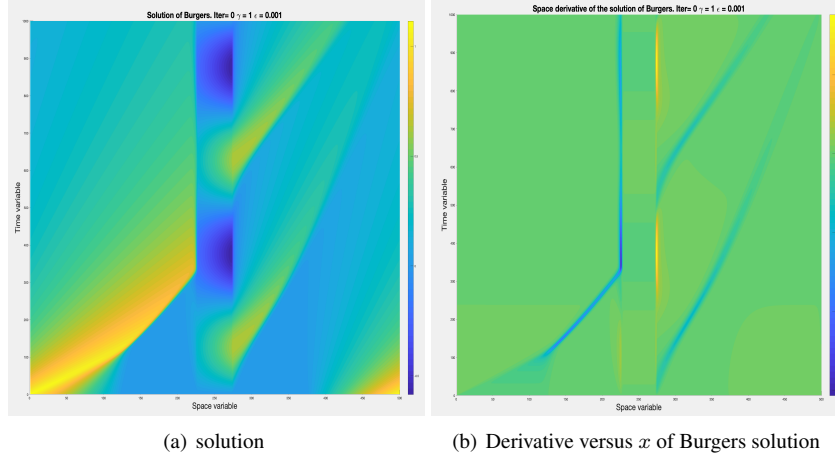
Figure 17: Burgers solution obtained without duality but a penalty term inside the structure (L^2).
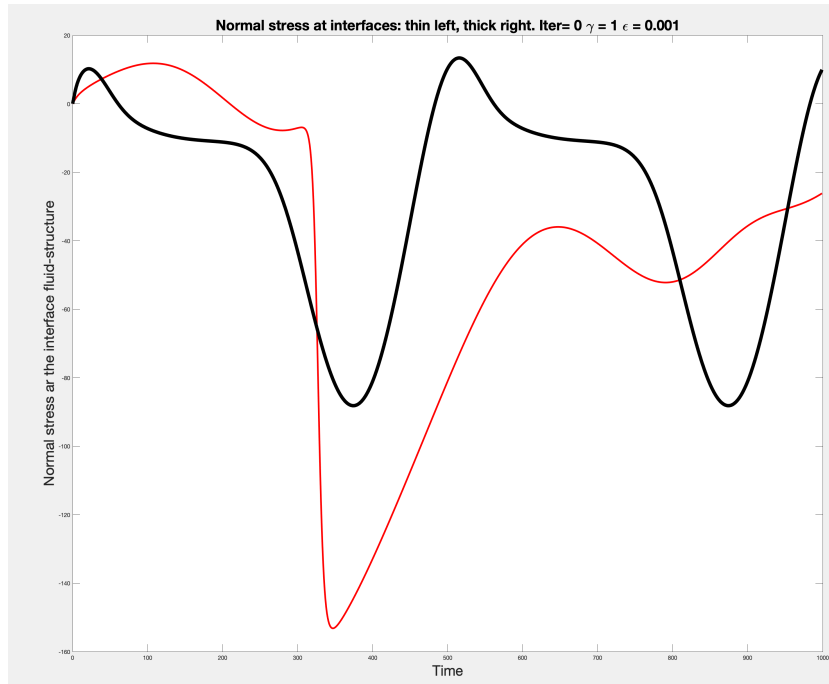
Figure 18: Normal stress at the f-s interface (no duality but a penalty term inside the structure.

(a) solution

(b) Derivative versus x of Burgers solution
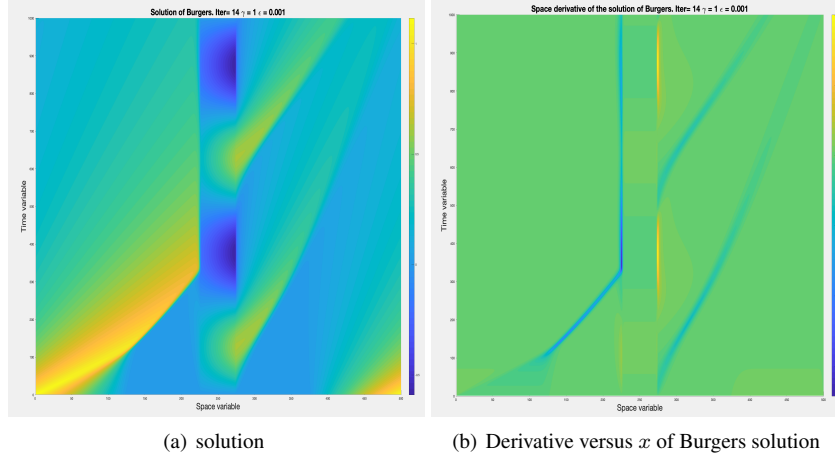
Figure 19: Burgers solution obtained with duality and a penalty term inside the structure.

(a) Normal stress at the f-s interface

(b) Lagrange multipliers at the f-s interfaces
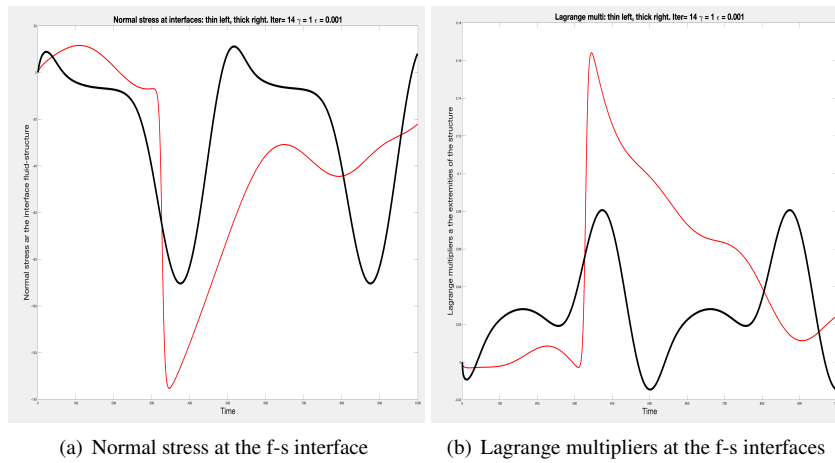
Figure 20: Normal stress and Lagrange multipliers at the f-s interface with duality and a penalty term inside the structure.

(a) Reference solution

(b) Derivative versus x of the reference solution
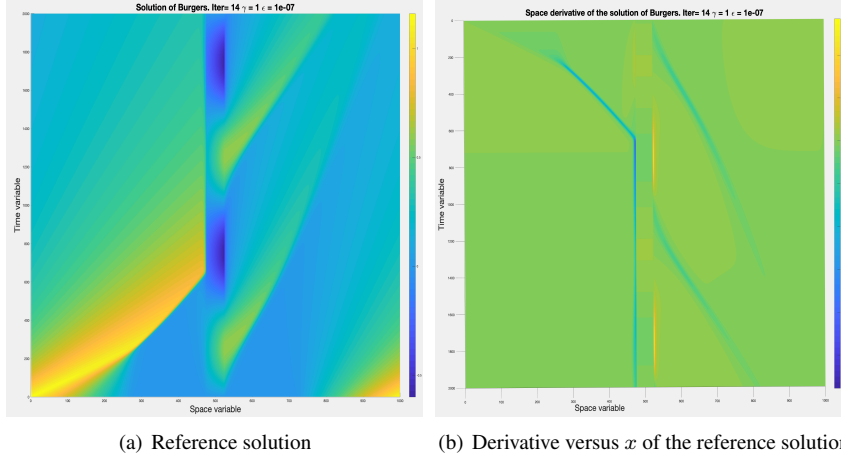
Figure 21: Reference solution of Burgers equation (duality + penalty term inside the structure + refined mesh + time step)

(a) Normal stress at the f-s interfaces

(b) Lagrange multipliers at the f-s interfaces, (the sign of the Lagrange multiplier is opposite to the one of the normal stress in this case)
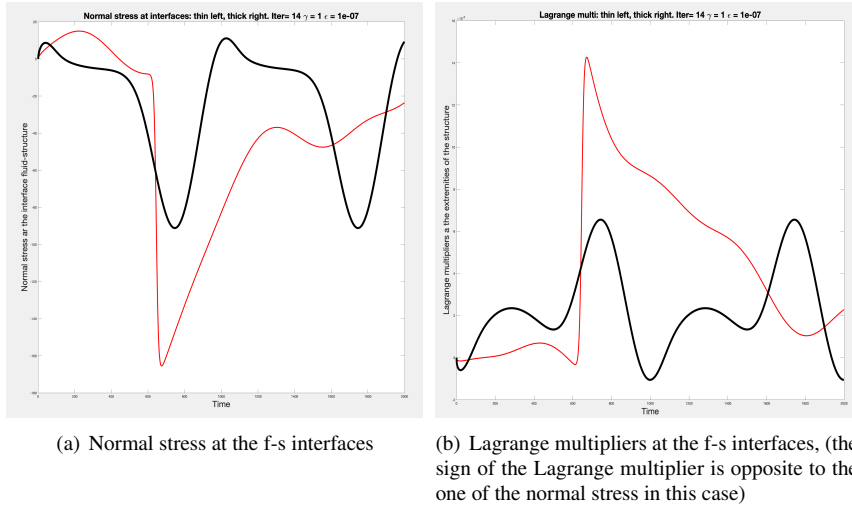
Figure 22: Reference solution (Figure 21(a)) obtained with duality and a penalty term inside the structure.

8.4 A two dimensional example

Let us consider a rectangular domain Ω containing a disc which is occupied by the structure. This is typically the case of a two dimensional problem representing the flow around a cylinder of radius $R = .1$ (see Figure 23). On the upper and lower boundaries the velocity is prescribed to zero. At the inlet (left part on Figure 24(a)), the velocity is uniform equal to $U = 1$ (excepted at the junction with the upper and lower boundaries). On the outlet of the boundary (on the right), the flow is assumed to be free and the normal stress vector is equal to zero. Obviously this is just for an example and it has no real influence on the evaluation of the methods used.
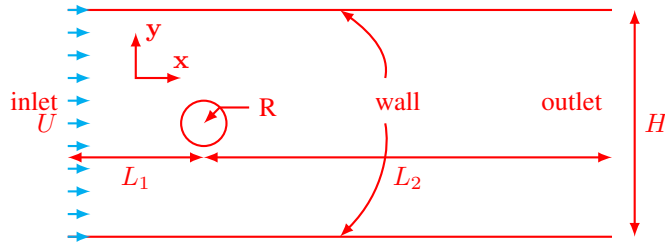
Figure 23: Imposed displacement of a cylinder in a transverse flow

The kinematical viscosity is $\nu = 4.10^{-3}$ and the mass density is $\varrho = 1$. The equations of the model are those of (incompressible) Stokes model explicited at section 2. The Reynolds number is about $Re = U2R/\nu = 50$, (the Strouhal frequency ((just as an indicator because Strouhal instabilities involves necessarily the convection term) is $f_S = .2U/2R = 1$ Hz and the reduced frequency (without dimension number) is $f_r = 2R/(TU) = .4 < 1$ (where T is the period of oscillation of the structure) which allows to use a quasi-static approximation (ie. the structure remains the same place for the fluid computations, see [17]).

The classical mini-element (velocity $P1$ $continuous$ $\oplus$ $bubble$ for each component and $P1$ continuous for the pressure, see [25]) has been used coupled with an Euler backward scheme for the time approximation. The Lagrange multiplier is defined only at the node of the structural boundary. The penalty parameter ε is always equal to .1.

A first computation leads to the stationary velocity field represented on Figures 24. The velocity field is on the left of Figure 24 and the pressure field on the right. These computation have been performed using the penalty-duality algorithm. One can observe that the boundary of the disc is perfectly represented on both graphics. A computation using the Stokes model but with Dirichlet boundary conditions on the disc (reference for comparison) leads exactly to the same solution for both the velocity and the pressure.

Then we did a similar computation but with only the penalty term on the surface of the disc ($\alpha = \beta = 0$ and $\gamma = 1$; see section 2). In fact it corresponds to case where there is no duality iteration. The results which should be compared to those of Figure 24 are plotted on Figure 25. One can see that the results are mainly different for the pressure (Figures on the right). Concerning the velocity the difference is less but exists. We have also represented the two component of the normal stress vector along the boundary of the

(a) Velocity magnitude

(b) Pressure field

Figure 24: Captures of the stationary fluid flow obtained with the penalty-duality algorithm

(a) Velocity magnitude

(b) Pressure field
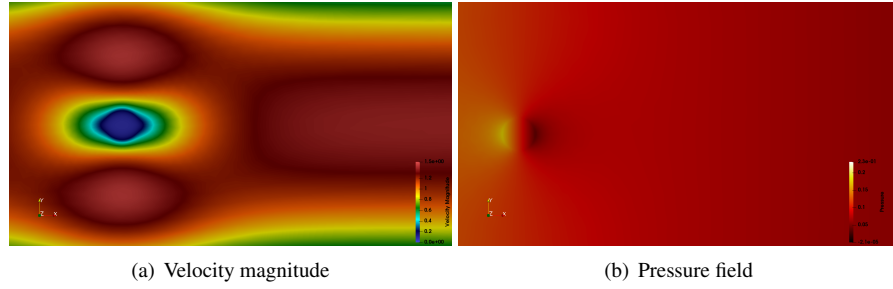
Figure 25: Captures of the stationary fluid flow obtained with only surface penalization ($\alpha = 0$, $\beta = 0$ and $\gamma = 1$)
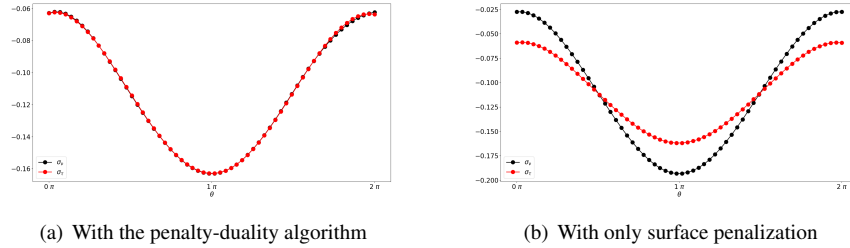

(a) With the penalty-duality algorithm

(b) With only surface penalization

Figure 26: σ_n and σ_t versus θ staring from the rear point of the circle up to the one corresponding to the stagnation point .

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2. The results for the second component of the velocity field are plotted at two instants in quadrature (t and $t + T/4$), on Figures 27(a) for the penalty method and on 27(b) for the penalty-duality algorithm. The difference appears clearly on the velocity field (second component which is more interesting in this computational case).
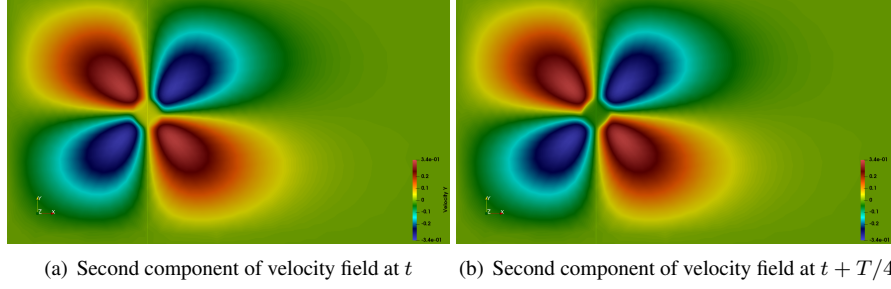

(a) Second component of velocity field at t

(b) Second component of velocity field at $t + T/4$

Figure 27: Results with only surface penalization $\alpha = 0$, $\beta = 0$ and $\gamma = 1$

(a) Second component of velocity field at t

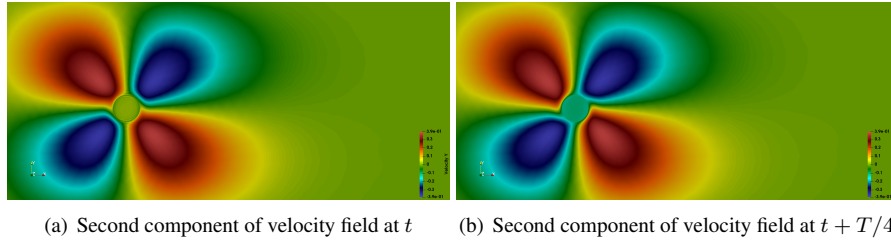(b) Second component of velocity field at $t + T/4$

Figure 28: Results with penalty-duality model

We have also plotted the two components of the normal stress vector on Figures 29. Here again the results obtained by the penalty-duality algorithm (on the left graphic) are much closer than those obtained when using only the penalty algorithm, to those obtained when using a Stokes solver with non-homogeneous boundary conditions (considered as the reference computation).
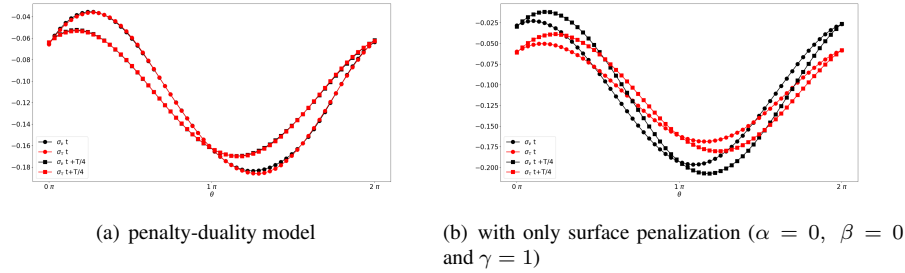

(a) penalty-duality model

(b) with only surface penalization ($\alpha = 0$, $\beta = 0$ and $\gamma = 1$)

Figure 29: Normal and tangentiel stress at t and $t + T/4$

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The use of the penalty-duality method improves the fluid-solid interface approximation. While on Figures 27 the fluid-solid interface is not clearly estimated, the cylinder is well drawn in Figures 28. The effect of the duality is mainly on the normal stress at the interface between the fluid and the structure. One can see by comparing Figures 26 and 29 that the duality improves the estimation on these quantities. Furthermore it is not necessary to use a very small value for the penalty parameter ε in order to obtain an accurate estimate on these normal stress components. This is a point to be underlined from these numerical tests which agrees with the theoretical study.

9   Conclusion

In this paper we have first studied several penalty methods in a simple fluid-structure model assuming the case of a small reduce frequency in order to partially decouple the inertia terms between the fluid and the structure. We discussed the possibilities of several penalty terms inside the structure (L^2 or/and H^1) norms but also by penalizing the continuity of the velocities at the boundary between the fluid and the structure.
Our goal has been to focus on the mathematical difficulties which can occur in the numerical schemes due to the ill conditioning of the penalty models.We compare the methods from the theoretical point of view and on a very simple 1D model. We also introduced in this framework the Bertsekas penalty-duality algorithm for the ensuring the velocity continuity at the fluid-structure interface. It appears that the penalty-duality algorithm applied to this quasi-static model without inertia terms is very efficient compared to other penalty methods. Furthermore it leads to a much better condition number of the numerical scheme as far as it not necessary to use a large value of the penalty coefficient.

In a second step we compared the various possibilities discussed in the first step on two very simple 1D fluid-structure models. The first one is the advection-diffusion equation with a prescribed movement of an immersed structure and the second one is a similar model but with Burgers equation for the fluid. In this case there a convection term (linear for the advection and non linear for Burgers equation). The numerical tests show that the L^2 penalty term inside the structures is very efficient as far as the penalty parameter is very small.

The coupling with the penalty-duality algorithm involving the velocity continuity at the interface enables to improve very slightly the numerical tests as far as the inner penalty parameter is small. But it is much more efficient if this parameter is more moderate. This can be therefore an interesting improvement in cases where the condition number is an important point in the numerical scheme. One can forecast that this is mainly the case for 2D and more for 3D models.
The last section 8.4 gives a two dimensional test on a simple incompressible flow around a cylinder for Stokes model. The results show that the interest of the penalty-duality method is meaningful in particular for the interface stress vector.

Because this paper is mainly theoretically oriented, we focused on the mathematical analysis of the penalty and penalty-duality methods for Stokes equations coupled with a rigid structure in movement. The tests are just an illustration in order to illustrate our purpose and to point out the limits of our conclusions. For a more physical study we refer to A. Falaise and E. Liberge [19].

Nevertheless, it is necessary to pursue the study for large displacements of the structure. One possibility is to use an embedding in a global Euler-Lagrange representation and it should be considered in future works. It will allow to treat large displacements of the structure due to the fluid. This aspect will be discussed in a forthcoming study [15] where we use the so-called θ-method [17] for transferring informations between the fluid model and the one of the structure which can be flexible and moving with large displacements inside the fluid.

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